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ATTITUDE PARAMETERIZATIONS AS HIGHER DIMENSIONAL MAP PROJECTIONS

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A generalization is proposed for a class of three-parameter attitude representations that are formulated as a product of the unit rotation vector and various functions of the rotation angle. When related to a four-dimensional unit quaternion, these three-dimensional representations are shown to be analogous to higher-dimensional azimuthal map projections from a three-dimensional unit sphere. Several types of these parameterizations are examined. Their relationships to the rotation matrix and their kinematics are derived. It is shown that kinematical passivity and optimality of the Rodrigues and modified Rodrigues parameters is a special case of the more general result that holds for a wider range of attitude representations. This result is used to formulate and compare passivity based control laws using various attitude representations.

INTRODUCTION

Connections between attitude parameterizations and higher-dimensional projections have been examined by several authors.[1,2,3] These examinations share a common realization that Euler parameters (also referred to as unit quaternions or simply quaternions in the context of this paper) reside on a three-dimensional unit sphere embedded in a four-dimensional space. For each location on this sphere there is a corresponding attitude and for each attitude there are two antipodal locations on the sphere. In Reference 1, Tsiotras establishes that a stereographic projection from the quaternion sphere onto a three-dimensional mapping hyperplane provides an elegant geometric interpretation for transforming quaternions into the Modified Rodrigues Parameters (MRPs). Schaub and Junkins in Reference 2, and later Mullen and Schaub in Reference 3 expand on this work by showing that projection point and mapping hyperplane can be manipulated to generate more general classes of attitude parameterizations called the Stereographic Orientation Parameters (SOPs) and the Hypersphere Stereographic Orientation Parameters (HSOPs) which include both the MRPs and the classical Rodrigues parameters (also referred to as Cayley-Rodrigues parameters) (CRPs).

This paper proposes a further generalization of the projection approach which considers all azimuthal projections (a group in which stereographic projections form a sub-group). For azimuthal projections points on a sphere are located using meridian and colatitude coordinates measured with respect to a reference pole. [4] A meridian is a great circle that passes through the pole and the point in question, and colatitude is a great arc angle along the meridian from the pole to the point in question. The pole is mapped as the origin onto the hyperplane and any other point is mapped so that its meridian coordinate is preserved and its distance from the origin is some function of its colatitude. For a two-dimensional sphere embedded in a three-dimensional space, meridian is fully defined by a single coordinate – its longitude or azimuth (hence, the name "azimuthal projections"). For a three-dimensional sphere embedded in a four-dimensional space, meridian must be defined by two coordinates or by three coordinates with a norm constraint. On the quater-

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nion sphere meridian coordinates define a direction of the spin axis that rotates the reference attitude (the pole) into the attitude in question. Colatitude on the quaternion sphere is simply half of the rotation angle about that spin axis. Hence, there is a straightforward analogy between formulations of map projections and attitude parameterizations.

This paper first introduces the projected rotation parameters (PRPs) and describes their relationship to rotation matrices and quaternions. The paper then defines the kinematics of PRPs and examines their passivity, optimality and nonlinearity. These results subsume those previously established for MRPs, CRPs and several other types of three-parameter attitude representations. Finally, the paper lists some commonly used azimuthal map projections and relates them to specific types of PRPs some of which are well known while others are new.

PROJECTED ROTATION PARAMETERS AS AZIMUTHAL PROJECTIONS FROM QUATERNION SPHERE

The standard azimuthal projection from a two-dimensional sphere onto a two-dimensional plane can be written as

$$x = (\cos a) r(\varphi), \; y = (\sin a) r(\varphi), \tag{1}$$

where x and y are the two planar map coordinates, φ is the colatitude and a is the azimuth (see Figure 1 for a simple example where $r(\varphi) = \tan\varphi$). Here r is some function of colatitude that defines the planar distance from the origin of a point projected onto the map. This function can be formulated in many different ways but in order to represent a well defined mapping it must satisfy the following mild restrictions: it must be monotonically increasing and it must pass through the origin. The first restriction ensures that larger colatitude angles correspond to larger distances on the map and that distinct colatitudes correspond to distinct distances on the map. This is important in order to be able to unambiguously deproject points from the planar map back onto the sphere. The second restriction ensures that there are no holes in the map and that the map's origin represents the reference pole of the sphere.

It is straightforward to extend this standard two-dimensional formulation to higher dimensions. Written in vector form, the following represents azimuthal projection from an N -dimensional sphere onto an N -dimensional hyperplane:

$$\mathbf{r} = \hat{\mathbf{n}} r(\varphi) \tag{2}$$

where $\hat{\mathbf{n}}$ is the N -dimensional unit vector that effectively defines the meridian using $N-1$ coordinates and r is still a function of colatitude that defines the distance from the origin of a point projected onto the hyperplane. Now consider a three-dimensional unit sphere embedded in a four-dimensional quaternion space. Let $\hat{\mathbf{q}}_0$ be the reference quaternion selected as the projection pole and let $\hat{\mathbf{q}}$ be some other unit quaternion. Both quaternions represent their respective attitudes. Rotation from $\hat{\mathbf{q}}_0$ to $\hat{\mathbf{q}}$ can be represented as another quaternion as[5]

$$\hat{\mathbf{Q}} = \hat{\mathbf{q}} \otimes \bar{\hat{\mathbf{q}}}_0 = \begin{bmatrix} \hat{\mathbf{n}} \sin\varphi & \cos\varphi \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{n}} \sin\dfrac{\phi}{2} & \cos\dfrac{\phi}{2} \end{bmatrix}^{\mathrm{T}}, \tag{3}$$

where $\bar{\hat{\mathbf{q}}}_0$ is a conjugate of $\hat{\mathbf{q}}_0$ and where $\otimes$ is a quaternion composition operator. Here $\hat{\mathbf{n}}$ represents the spin axis of any rotation that passes through attitudes defined by $\hat{\mathbf{q}}_0$ and $\hat{\mathbf{q}}$. On the quaternion sphere it defines the meridian that passes through the corresponding quaternion points. The colatitude φ, which is the angle between $\hat{\mathbf{q}}_0$ and $\hat{\mathbf{q}}$, is half of the angle of rotation ϕ about $\hat{\mathbf{n}}$ that rotates the attitude represented by $\hat{\mathbf{q}}_0$ into the attitude represented by $\hat{\mathbf{q}}$. The projected coordinates' relationship to attitude is more natural if they are formulated in terms of rotation angles rather than quaternion colatitudes on the quaternion sphere. So the PRPs are defined as

$$\mathbf{r} = \hat{\mathbf{n}} r(\phi) . \tag{4}$$

Here the projection function r is a monotonically increasing and continuously differentiable function of ϕ (at least within its domain between 0 and some maximum angle $\phi_{\max}$) that also passes through the origin, i.e. $r(0) = 0$.

In order to be practical as an attitude parameterization, the PRPs must have straightforward closed form transformations to and from quaternions and rotation matrices. It is clear from the definition of $\mathbf{r}$ that these transformations must involve $\hat{\mathbf{n}}$ and ϕ, whose relationships with quaternions and rotation matrices are well documented. Let $\mathbf{C}$ be the rotation matrix that corresponds to $\hat{\mathbf{Q}}$ as defined in Eq.(3), then[5]

$$\mathbf{C} = \exp([\![\hat{\mathbf{n}}]\!]\phi) = \mathbf{I} + \sin\phi [\![\hat{\mathbf{n}}]\!] + (1 - \cos\phi)[\![\hat{\mathbf{n}}]\!]^2 . \tag{5}$$

Here $[\![\mathbf{v}]\!]$ denotes a skew-symmetric matrix

$$[\![\mathbf{v}]\!] = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{6}$$

for any three-dimensional vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{I}$ denotes an identity matrix. Once $\hat{\mathbf{n}}$ and ϕ are obtained from either $\hat{\mathbf{Q}}$ or $\mathbf{C}$ using any one of the well-documented algorithms, $\mathbf{r}$ can be constructed directly from Eq. (4). The ease of the inverse transformation from $\mathbf{r}$ to either $\hat{\mathbf{Q}}$ or $\mathbf{C}$ hinges on the simplicity of the deprojection function r^{-1}. Note that because $\hat{\mathbf{Q}}$ and $\mathbf{C}$ are written in terms of trigonometric functions of ϕ, it may not be necessary to obtain ϕ explicitly. Instead, it may be possible to formulate both r and r^{-1} directly in terms of trigonometric functions of ϕ and thus greatly simplify relationships between $\mathbf{r}$ and $\hat{\mathbf{Q}}$ or $\mathbf{C}$. In fact, as shown later in this paper most well known attitude parameterizations benefit from such simplifications.

KINEMATICS OF PROJECTED ROTATION PARAMETERS

For dynamical modeling, it is also important that the PRPs possess simple kinematical relationships with the angular velocity vector. Let $\boldsymbol{\omega}$ denote the angular velocity vector in the rotating frame. Differential equation for $\dot{\mathbf{r}}$ can be constructed by a direct differentiation of Eq. (4)

$$\dot{\mathbf{r}} = \dot{\hat{\mathbf{n}}} r(\phi) + \hat{\mathbf{n}} r'(\phi)\dot{\phi} . \tag{7}$$

Here and throughout the paper "$'$" denotes partial differentiation of a function with respect to the specified argument. So in this case $r'(\phi)$ denotes differentiation of r with respect to ϕ. Differential equations for $\dot{\hat{\mathbf{n}}}$ and $\dot{\phi}$ are well-known[5] and can be substituted in Eq.(7) to ultimately obtain

$$\dot{\mathbf{r}} = \mathbf{G}(\mathbf{r})\boldsymbol{\omega}\,, \tag{8}$$

where

$$\mathbf{G}(\mathbf{r}) = r'(\phi)\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} - \frac{1}{2}[\![\mathbf{r}]\!] - [\![\hat{\mathbf{n}}]\!]^2 \frac{r(\phi)\sin\phi}{2\left(1-\cos\phi\right)}\,. \tag{9}$$

The inverse kinematics can be also obtained after some algebra:

$$\boldsymbol{\omega} = \mathbf{H}(\mathbf{r})\dot{\mathbf{r}}\,, \tag{10}$$

where

$$\mathbf{H}(\mathbf{r}) = \frac{1}{r'(\phi)}\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} - \frac{\sin\phi}{r(\phi)}[\![\hat{\mathbf{n}}]\!]^2 + \frac{1-\cos\phi}{r^2(\phi)}[\![\mathbf{r}]\!]\,. \tag{11}$$

Note that, as expected, the direct and inverse kinematics matrices, $\mathbf{G}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$, are inverses of each other. Note also that the relationship

$$\dot{\phi} = \hat{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\omega} \;\text{ with }\; \hat{\mathbf{n}} = \frac{\mathbf{r}}{r(\phi)} \tag{12}$$

provides an alternative to using the deprojection function r^{-1} for obtaining quaternions or rotation matrices from $\mathbf{r}$. If $\mathbf{r}$ is numerically propagated using Eq.(8) then ϕ, and thus effectively quaternions and rotation matrices, can be numerically propagated alongside using Eq.(12). This makes it possible to kinematically employ projections for which the inverse function is either unavailable or too costly to compute in closed form.

PASSIVITY OF PROJECTED ROTATION PARAMETERS

In References 6, Tsiotras describes the importance of using kinematically passive attitude parameterizations. If passivity can be established then simple closed-loop control laws with certain optimality properties can be employed. In particular, Tsiotras identifies passivity and optimality of the MRPs and CRPs.[6,7] This paper extends these results for all types of PRPs and demonstrates MRPs and CRPs fit within the more general framework. Following Tsiotras, consider the inner product of the input and output of the attitude kinematics. Note from Eq.(9) that

$$\mathbf{G}(\mathbf{r})\mathbf{r} = \mathbf{G}^{\mathrm{T}}(\mathbf{r})\mathbf{r} = r'(\phi)\mathbf{r}\,. \tag{13}$$

In other words, regardless of the choice of the projection function r, the vector $\mathbf{r}$ of the PRPs is both the left- and the right- eigenvector of the matrix $\mathbf{G}(\mathbf{r})$ and has the associated eigenvalue $r'(\phi)$. Note that this eigenvalue is positive by construction because the projection function is required to be a monotonically increasing function of ϕ. Using Eqs.(8) and (13), the inner product of $\mathbf{r}$ and $\dot{\mathbf{r}}$ can be related to the inner product of $\mathbf{r}$ and $\boldsymbol{\omega}$:

$$\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}} = \mathbf{r}^{\mathrm{T}}\mathbf{G}(\mathbf{r})\boldsymbol{\omega} = r'(\phi)\mathbf{r}^{\mathrm{T}}\boldsymbol{\omega}\,. \tag{14}$$

The left-hand side of this equation can be replaced by

$$\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}} = r(\phi)\dot{r}(\phi) = r(\phi)r'(\phi)\dot{\phi}\,, \tag{15}$$

which leads to

$$r(\phi)r'(\phi)\dot{\phi} = r'(\phi)\mathbf{r}^{\mathrm{T}}\boldsymbol{\omega}\,. \tag{16}$$

Here both sides can be divided by $r'(\phi)$ because by construction $r'(\phi) > 0$ and is, therefore, non-zero. The resulting expression

$$r(\phi)\dot{\phi} = \mathbf{r}^{\mathrm{T}}\boldsymbol{\omega} \tag{17}$$

can be used to establish passivity of the attitude kinematics. Consider the following storage function

$$V(\mathbf{r}) = \int_0^{\phi} r(\xi)d\xi\,, \tag{18}$$

which is guaranteed to be a positive definite function of $\mathbf{r}$ given the restrictions placed on the projection function r. Its derivative along the kinematical trajectories of $\mathbf{r}$ is reduced to the following form by taking advantage of the relationship in Eq. (17):

$$\dot{V}(\mathbf{r}) = r(\phi)\dot{\phi} = \mathbf{r}^{\mathrm{T}}\boldsymbol{\omega}\,. \tag{19}$$

The fact that the derivative of the storage function is exactly equal to the inner product of the input $\boldsymbol{\omega}$ and the output $\mathbf{r}$ of the dynamical system described by Eq.(8) indicates that this system is lossless (which is a particular type of passivity).[6,8] In Reference 6 Tsiotras demonstrates that the kinematics of MRPs and CRPs are lossless by finding appropriate storage functions. This work uncovers a simple recipe for generating such storage functions for any type of PRPs examples of which are shown later in this paper. Of course, the proof described by Eqs.(18) and (19) makes it unnecessary to generate storage functions to verify passivity of individual types of PRPs: it shows that kinematics of any and all types of PRPs are lossless.

This result has important implications for designing attitude control laws. Since kinematical passivity of PRPs has been established, it is straightforward to show that the following linear closed-loop control law

$$\boldsymbol{\tau} = -k_r\mathbf{r} - k_\omega\boldsymbol{\omega} \tag{20}$$

must (almost) globally asymptotically stabilize the dynamical system consisting of a cascade interconnection of the rigid-body dynamics and PRPs kinematics. Here $\boldsymbol{\tau}$ is the external torque generated by the control law, and $k_r > 0$ and $k_\omega > 0$ are two positive constants. The proof is entirely analogous to proofs put forward by Tsiotras for the MRPs and CRPs. See Reference 6 for further details. Note that selecting various projection functions in Eq.(4) opens new opportunities for shaping performance of the closed-loop attitude control.

OPTIMALITY OF PROJECTED ROTATION PARAMETERS

Consider now optimality of the closed-loop control using PRPs. In a limited sense proposed by Tsiotras for the MRPs and CRPs in Reference 9, kinematical optimality can be established using the following quadratic performance index[9]

$$J(\mathbf{r}_0, \boldsymbol{\omega}) = \frac{1}{2}\int_0^{\infty}\left\{k_r^2\mathbf{r}^{\mathrm{T}}\mathbf{r} + \boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\omega}\right\}dt \tag{21}$$

According to Hamilton-Jacobi theory, the optimal feedback $\boldsymbol{\omega}^*$ must satisfy

$$0 = \min_{\boldsymbol{\omega}} \left\{ \frac{1}{2} k_r^2 \mathbf{r}^{\mathrm{T}} \mathbf{r} + \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{\omega} + \frac{\partial W}{\partial \mathbf{r}} \mathbf{G}(\mathbf{r}) \boldsymbol{\omega} \right\} \tag{22}$$

from which

$$\boldsymbol{\omega}^*(\mathbf{r}) = -\mathbf{G}^{\mathrm{T}}(\mathbf{r}) \left(\frac{\partial W}{\partial \mathbf{r}} \right)^{\mathrm{T}} \tag{23}$$

where $W(\mathbf{r})$ is a positive definite value function. Let $W(\mathbf{r}) = k_r V(\mathbf{r})$, then, using Eq.(18), $W(\mathbf{r})$ can be expressed as

$$W(\mathbf{r}) = k_r \int_0^{\phi} r(\xi) d\xi = k_r \int_{r(0)}^{r(\phi)} \frac{r(\xi)}{r'(\xi)} dr = k_r \int_{\mathbf{r}(0)}^{\mathbf{r}(\phi)} \frac{r(\xi)}{r'(\xi)} \hat{\mathbf{n}}^{\mathrm{T}} d\mathbf{r} \tag{24}$$

so that it is easily differentiated with respect to $\mathbf{r}$:

$$\frac{\partial W}{\partial \mathbf{r}} = k_r \frac{r(\phi)}{r'(\phi)} \hat{\mathbf{n}}^{\mathrm{T}}. \tag{25}$$

The result of this differentiation can be substituted into Eq.(23) and, with the aid of Eq.(13), reduced to

$$\boldsymbol{\omega}^*(\mathbf{r}) = -k_r \frac{r(\phi)}{r'(\phi)} \mathbf{G}^{\mathrm{T}}(\mathbf{r}) \hat{\mathbf{n}} = -k_r \mathbf{r}. \tag{26}$$

Therefore, the choice of $W(\mathbf{r}) = k_r V(\mathbf{r})$ satisfies the optimality condition and $\boldsymbol{\omega}^*(\mathbf{r})$ is the optimal control for which the optimal cost is given by

$$J^*(\mathbf{r}_0) = J(\mathbf{r}_0, \boldsymbol{\omega}^*) = k_r V_r(\mathbf{r}_0). \tag{27}$$

These results demonstrate that the optimality established by Tsiotras for the MRPs and CRPs extends to any type of PRPs.

LINEARITY OF PROJECTED ROTATION PARAMETERS

Consider attitude stabilization using the PRPs with the kinematically optimal control law shown in Eq.(20). For a rigid body with the unit inertia matrix starting from rest the following differential equation describes the closed loop evolution of the rotation angle:

$$\ddot{\phi} = -k_r r - k_\omega \dot{\phi} \text{ with } \phi_0 = \phi(t_0). \tag{28}$$

The corresponding state error transition matrix for $\begin{bmatrix} \Delta\phi & \Delta\dot{\phi} \end{bmatrix}^{\mathrm{T}}$ evolves according to

$$\dot{\mathbf{\Phi}}(t, t_0) = \begin{bmatrix} 0 & 1 \\ -k_r r'(\phi) & -k_\omega \end{bmatrix} \mathbf{\Phi}(t, t_0) \text{ with } \mathbf{\Phi}(t_0, t_0) = \mathbf{I}. \tag{29}$$

Let $\pm\Delta\phi_0$ be the expected (or the largest) perturbation of the initial rotation angle ϕ_0. Then the following nonlinearity index proposed by Junkins in Reference, and by Junkins and Singla in Reference can be computed for the closed-loop system:

$$\nu_\phi(t,t_0) = \sup_{\phi\pm\Delta\phi} \frac{\left\| \mathbf{\Phi}_{\pm\Delta\phi}(t,t_0) - \mathbf{\Phi}(t,t_0) \right\|_F}{\left\| \mathbf{\Phi}(t,t_0) \right\|_F} . \tag{30}$$

Here subscript $\pm\Delta\phi$ indicates perturbed trajectories starting with $\phi_0 \pm \Delta\phi_0$. This index can be used to evaluate nonlinearity of the attitude error propagation using any type of the PRPs.

EXAMPLES OF PROJECTED ROTATION PARAMETERS

As stated previously, the fact that the projection function r is subject to only mild restrictions permits many types of the PRPs. Consider first the Higher Order Rodrigues Parameters (HORPs) introduced by Tsiotras, Junkins and Schaub in Reference 10 as a generalization of the MRPs and CRPs. When examined as a sub-group within the PRPs, the HORPs provide interesting insights and exhibit relationships that are useful in subsequent derivations for other types of the PRPs.

Higher Order Rodrigues Parameters

In the context of this paper, the HORPs are analogous to azimuthal projections for which projection functions are given by

$$r(\phi) = \tan\frac{\phi}{2m} \text{ with } m = 1,2,3,4,... \tag{31}$$

Setting $m=1$ produces the CRPs, $m=2$ produces the MRPs, and the higher integer values of m produce the corresponding higher orders of the HORPs. Let $\boldsymbol{\rho}_m$ denote the mth order HORPs defined according to Eq.(4) with the projection function from Eq.(31). Then the corresponding rotation matrix can be concisely written in terms of the higher order Cayley transforms:[10]

$$\mathbf{C} = \left(\mathbf{I} + [\![\boldsymbol{\rho}_m]\!]\right)^m \left(\mathbf{I} - [\![\boldsymbol{\rho}_m]\!]\right)^{-m} . \tag{32}$$

The direct and inverse kinematics for $\boldsymbol{\rho}_m$ can be derived from Eqs.(9) and (11), respectively. In these equations trigonometric terms equal to $\tan\frac{\phi}{2}$ can be recognized as ρ - the magnitude of the CRPs. With the appropriate substitution of ρ, the direct kinematics matrix for $\boldsymbol{\rho}_m$ becomes

$$\mathbf{G}(\boldsymbol{\rho}_m) = \frac{1+\rho_m^2}{2m\rho_m^2}\boldsymbol{\rho}_m\boldsymbol{\rho}_m^{\mathrm{T}} - \frac{1}{2}[\![\boldsymbol{\rho}_m]\!] - \frac{[\![\boldsymbol{\rho}_m]\!]^2}{2\rho\rho_m} = \left[\frac{1+\rho_m^2}{2m\rho_m^2} - \frac{1}{2\rho\rho_m}\right]\boldsymbol{\rho}_m\boldsymbol{\rho}_m^{\mathrm{T}} - \frac{1}{2}[\![\boldsymbol{\rho}_m]\!] + \mathbf{I}\frac{\rho_m}{2\rho} \tag{33}$$

and the inverse kinematics matrix for $\boldsymbol{\rho}_m$ becomes

$$\mathbf{H}(\boldsymbol{\rho}_m) = \frac{2m}{\rho_m^2\left(1+\rho_m^2\right)}\boldsymbol{\rho}_m\boldsymbol{\rho}_m^{\mathrm{T}} - \frac{2\rho}{\rho_m^3\left(1+\rho^2\right)}[\![\boldsymbol{\rho}_m]\!]^2 + \frac{2\rho^2}{\rho_m^2\left(1+\rho^2\right)}[\![\boldsymbol{\rho}_m]\!] . \tag{34}$$

In order to complete these expressions solely in terms of $\boldsymbol{\rho}_m$, consider how ρ can be expressed as a function of ρ_m. This can be done via the following identity for tangents of multiple angles:

$$\rho = \tan\frac{\phi}{2} = \tan m\frac{\phi}{2m} = i\frac{\left(1-i\tan\frac{\phi}{2m}\right)^m - \left(1+i\tan\frac{\phi}{2m}\right)^m}{\left(1-i\tan\frac{\phi}{2m}\right)^m + \left(1+i\tan\frac{\phi}{2m}\right)^m} = i\frac{\left(1-i\rho_m\right)^m - \left(1+i\rho_m\right)^m}{\left(1-i\rho_m\right)^m + \left(1+i\rho_m\right)^m} \qquad (35)$$

Derived on this identity, Table 1 lists expressions for ρ as a function of ρ_m for various orders.

Table 1. Relationship between Magnitudes of CRPs and Various Orders of HORPs.

m	$\rho(\rho_m)$
1 (CRPs)	$\rho = \rho_1$
2 (MRPs)	$\rho = \dfrac{2\sigma}{1-\sigma^2}$, $\sigma = \rho_2$
3	$\rho = \dfrac{\rho_3\left(\rho_3^2-3\right)}{3\rho_3^2-1}$
4	$\rho = \dfrac{4\rho_4\left(\rho_4^2-1\right)}{\rho_4^4-6\rho_4^2+1}$
5	$\rho = \dfrac{\rho_5\left(\rho_5^4-10\rho_5^2+5\right)}{5\rho_5^2\left(\rho_5^2-2\right)+1}$
∞	$\rho = \lim\limits_{m\to\infty}\left[i\dfrac{\left(1-i\rho_m\right)^m-\left(1+i\rho_m\right)^m}{\left(1-i\rho_m\right)^m+\left(1+i\rho_m\right)^m}\right] = i\dfrac{1-\exp i\phi}{1+\exp i\phi} = \tan\dfrac{\phi}{2}$

The last row in this table simply illustrates how the relationship in Eq.(35) can be taken to its limit. It is easy to verify that for $m = 1, 2$ substituting results from this table into Eqs.(33) and (34) recovers well-known kinematical expressions for the CRPs and MRPs, respectively (see, for example, Reference 5). The same is true for the higher order parameterizations using $m = 3, 4$ whose kinematical expressions can be compared to those listed in Reference 10.

It is instructive to confirm the passivity of the HORPs by showing how their storage functions follow from Eq.(18). Substituting Eq.(31) into Eq.(18) and simplifying yields

$$V(\boldsymbol{\rho}_m) = \int_0^\phi \tan\frac{\xi}{2m}d\xi = -2m\ln\cos\frac{\phi}{2m} = m\ln\left(1+\tan^2\frac{\phi}{2m}\right) = m\ln\left(1+\boldsymbol{\rho}_m^{\mathrm{T}}\boldsymbol{\rho}_m\right). \qquad (36)$$

It is trivial to verify that when $m = 1, 2$ this simple general expression matches the well-known storage functions found by Tsiotras for the CRPs and MRPs.[6]

The nonlinearity index based on Eqs.(29) and (30) requires $r'(\phi)$ which for the HORPs is easily computed by differentiating Eq.(31):

$$r'(\phi) = \frac{1}{2m}\left(1 + \tan^2 \frac{\phi}{2m}\right) = \frac{1}{2m}\left(1 + r^2(\phi)\right). \tag{37}$$

To summarize this section of the paper: the HORPs are particular types of the PRPs which afford elegant transformations to and from rotation matrices via the higher order Cayley transforms. Their kinematics are passive and are easily derived within the general framework of the PRPs (although the derivations become increasingly tedious at higher orders). These results are used in the next section where additional attitude parameterizations are derived from various azimuthal projections. It is shown that these parameterizations can be related to the HORPs which simplifies many of the derivations.

Standard Azimuthal Map Projections

Given the analogy between azimuthal map projections and PRPs, it is natural to consider how some of the commonly used projections can be interpreted as attitude parameterizations. Note that many map projections include constant scaling factors designed to locally preserve distances or areas which is usually not a concern with attitude parameterizations. Also, note that azimuthal map projections use colatitude as one of their coordinates whereas attitude parameterizations are formulated using rotation angle. Therefore, correspondence between map projections and attitude parameterizations is established to within some constant scale factor and the angles involved in attitude parameterizations are halved compared to angles involved in map projections.

Some of the most common azimuthal map projections are included in Table 2 but the list is by no means exhaustive.

Table 2. Azimuthal Projection – Attitude Parameterization Correspondence.

Azimuthal Projection [4]	Attitude Parameterization*, $\mathbf{r} = \hat{\mathbf{n}} r(\phi)$	$r(\phi)$	$r'(\phi)$
Azimuthal Equidistant	Equidistant Orientation Parameters (EOPs) (Rotation or Euler Vector), $\boldsymbol{\phi}$	ϕ	1
Gnomonic	CRPs, $\boldsymbol{\rho}$	$\tan\frac{\phi}{2}$	$\frac{1}{2}\left(1 + \tan^2\frac{\phi}{2}\right)$
Stereographic	MRPs, $\boldsymbol{\sigma}$	$\tan\frac{\phi}{4}$	$\frac{1}{4}\left(1 + \tan^2\frac{\phi}{4}\right)$
Orthographic	Orthographic Parameters (OPs) (Quaternion Vector Part), $\boldsymbol{\eta}$	$\sin\frac{\phi}{2}$	$\frac{1}{2}\cos\frac{\phi}{2}$

* Names of new attitude parameterizations are italicized.

Lambert Azimuthal Equal Area	*Lambert Parameters (LPs),* $\boldsymbol{\lambda}$	$\sin\frac{\phi}{4}$	$\frac{1}{4}\cos\frac{\phi}{4}$
Breusing Geometric	*Breusing Parameters (BPs),* $\boldsymbol{\beta}$	$\tan\frac{\phi}{4}\sqrt{\cos\frac{\phi}{4}}$	$\frac{3+\cos\frac{\phi}{2}}{16\sqrt{\cos^3\frac{\phi}{4}}}$
Negative Perspective (D – distance from the center measured away from the pole)*	*Negative Perspective Parameters (NPPs),* $\mathbf{p}_-$	$\frac{(D+1)\sin\frac{\phi}{2}}{D+\cos\frac{\phi}{2}}$	$\frac{(D+1)\left(D\cos\frac{\phi}{2}+1\right)}{2\left(D+\cos\frac{\phi}{2}\right)^2}$
Positive Perspective (D – distance from the center measured toward the pole)	*Positive Perspective Parameters (PPPs),* $\mathbf{p}_+$	$\frac{(D-1)\sin\frac{\phi}{2}}{D-\cos\frac{\phi}{2}}$	$\frac{(D-1)\left(D\cos\frac{\phi}{2}-1\right)}{2\left(D-\cos\frac{\phi}{2}\right)^2}$

As stated previously, the advantage of connecting various types of the PRPs to the HORPs lies in the simplicity of the latter's relationship with the corresponding rotation matrix. Indeed, if for some order m the expression for $\boldsymbol{\rho}_m$ in terms of $\mathbf{r}$ is known then substituting $\boldsymbol{\rho}_m(\mathbf{r})$ directly into Eq.(32) produces an elegant expression for $\mathbf{C}$ effectively in terms of $\mathbf{r}$. If in addition the differential relationships between $\boldsymbol{\rho}_m$ and $\mathbf{r}$ are known then both the direct and inverse kinematics can be easily derived from Eqs.(8) and (10). The direct kinematics can be recast using $\mathbf{r}'(\boldsymbol{\rho}_m)$ as

$$\dot{\mathbf{r}} = \mathbf{r}'(\boldsymbol{\rho}_m)\dot{\boldsymbol{\rho}}_m = \mathbf{r}'(\boldsymbol{\rho}_m)\mathbf{G}(\boldsymbol{\rho}_m)\boldsymbol{\omega} \tag{38}$$

and the inverse kinematics – using $\boldsymbol{\rho}'_m(\mathbf{r})$ as

$$\boldsymbol{\omega} = \mathbf{H}(\boldsymbol{\rho}_m)\dot{\boldsymbol{\rho}}_m = \mathbf{H}(\boldsymbol{\rho}_m)\boldsymbol{\rho}'_m(\mathbf{r})\dot{\mathbf{r}}\,. \tag{39}$$

Of course, the same expressions can be computed directly from equations that use $\hat{\mathbf{n}}$ and ϕ without first relating $\mathbf{r}$ to $\boldsymbol{\rho}_m$. Which approach is easier depends solely on a parameterization. All of the relevant expressions for the types of the PRPs introduced in this paper are presented in details in the Appendix.

Mercator Map Projections

The original interpretation of PRPs is based on azimuthal projections but it is possible to also adopt projection functions from other types of projections. An interesting example is the Gudermannian function which is used to convert between the latitudinal distance on the Mercator projection and the arc length on the sphere. The function, denoted gd, relates circular and hyperbolic trigonometric functions without using complex numbers. There are several formal definitions of this function some of which are listed below:

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* Selecting various distances for this projection can lead to gnomonic, stereographic and orthographic projections.

$$gd(r) = \arcsin\tanh(r) = \arctan\sinh(r) = 2\arctan\tanh\frac{r}{2} = \int_0^r \text{sech}(t)dt\ . \tag{40}$$

The inverse Gudermannian function is also well defined and has several forms some of which are listed below:

$$gd^{-1}(\phi) = \text{arcsinh}\tan(\phi) = \text{arctanh}\sin(\phi) = 2\,\text{arctanh}\tan\frac{\phi}{2} = \int_0^\phi \sec(\xi)d\xi\ . \tag{41}$$

As a side note, consider an intriguing similarity between the Gudermannian relationship

$$\tanh\frac{gd^{-1}(\phi)}{2} = \tan\frac{\phi}{2} \tag{42}$$

and another relationship uncovered by Tsiotras when examining the higher order Cayley transforms[11]

$$\tanh[\![\hat{\mathbf{n}}\phi]\!] = [\![\hat{\mathbf{n}}]\!]\tan\phi\ . \tag{43}$$

Although the Mercator projector is cylindrical and not azimuthal, it is possible to adopt its Gudermannian, or more precisely, its inverse Gudermannian mapping as the projection function for new types of the PRPs. Consider a family of projection functions based on Eq.(41):

$$r(\phi) = gd^{-1}(\frac{\phi}{m}) = 2\,\text{arctanh}\tan\frac{\phi}{2m} = 2\,\text{arctanh}\,(\rho_m(\phi)) \ \text{ with } \ m = 1,2,3,4,... \tag{44}$$

By construction these projections are related via hyperbolic tangent to the projections used by the HORPs. Let $\boldsymbol{\mu}_m$ denote the mth order Mercator parameters (MPs) defined using the mth order projection function from Eq.(44). Then

$$\mu_m = 2\,\text{arctanh}\,(\rho_m) \tag{45}$$

and

$$\rho_m = \tanh\frac{\mu_m}{2} \tag{46}$$

which provides an easy connection $\boldsymbol{\rho}_m$ so that once again the elegant relationship to the rotation matrix $\mathbf{C}$ shown in Eq.(32) can be employed. The differential relationships between $\boldsymbol{\mu}_m$ and $\boldsymbol{\rho}_m$ are also straight-forward. They, along with all the other relevant expressions involving $\boldsymbol{\mu}_m$, are listed in the Appendix.

To summarize, projection of the rotation angle via the Mercator projection function is equivalent to a two-step projection in which the first step generates the CRPs and the second step projects them via the inverse hyperbolic tangent. The same relationship is maintained between the higher orders of the MPs and the corresponding orders of the HORPs. Note that $\boldsymbol{\mu}_m$ suffers from singularity when $\rho_m^2 = 1$ which effectively halves the domain of acceptable rotation angles ϕ: it changes from being less than $m\pi$ for $\boldsymbol{\rho}_m$ to being less than $\frac{m\pi}{2}$ for $\mathbf{\mu}_m$. For example, in order to be applicable within the same domain as the CRPs, the MPs must be second order and, therefore, be derived from the MRPs.

ATTITUDE CONTROL EXAMPLES USING PROJECTED ROTATION PARAMETERS

Consider how performance of the kinematically optimal control based on Eq.(20) differs when it is formulated using different types of PRPs. For all types, assume that the rigid body has the unit inertia matrix and that it starts at rest with the initial rotation angle $\phi_0 = 170$ deg. Set $k_w = 1$ and let k_r be selected for each type such that the initial rotational acceleration is the same: $\ddot{\phi}_0 = -10$ deg/s^2. For each type, propagate for 2 min both the state and the state error transition matrix and do it both for the nominal trajectory and for the perturbed trajectories starting with $\pm\Delta\phi_0 = \pm 0.25$ deg. Resulting time histories for $\phi(t)$, $\dot{\phi}(t)$, $\ddot{\phi}(t)$ and $\nu(t,t_0)$ are shown in Figures 2-5 for the following types of PRPs: OPs, CRPs, MRPs, EOPs, LPs, 3rd order HORPs, 4th order HORPs and 2nd order MPs. Figure 2 demonstrates that given the same limited control authority the CRPs and the 2nd order MPs significantly underperform compared to the other parameterizations. While rotation angle trajectories for the remaining parameterizations are generally close, the LPs outperform every other type of the PRPs by a significant margin. For example, the rotation angle falls below 5 deg after about 53 seconds when using the LPs but it takes about 70 seconds to accomplish the same when using the MRPs. The rotational rate trajectories shown in Figure 3 generally follow the same pattern as the rotation angle trajectories in Figure 2: results for the CRPs and the 2nd order MPs are significantly different from the other parameterizations. The 2nd order MPs and particularly the CRPs generate rotational rates that do not peak initially as high as those for other parameterizations which causes the rotation angle trajectories to descent slower and ultimately leads to poorer control performances for the CRPs and the 2nd order MPs. As designed using the appropriate selection of the control gain k_r, all of the rotational acceleration trajectories shown in Figure 4 start at the same value of $\ddot{\phi}_0 = -10$ deg/s^2. Generally, all trajectories remain clustered together, although as before, results for the CRPs and the 2nd order MPs are more distinct compared to the rest. These parameterizations cause faster declines in the magnitude of the rotational acceleration and also produce higher overshoots. It is interesting that relatively small changes exhibited in the control trajectories cause significant differences in the closed-loop performance. The CRPs and the 2nd order MPs also exhibit the highest nonlinearities among the tested parameterizations (see Figure 5). The next highest (although much lower) nonlinearities are shown by the LPs and the MRPs. The rest of the tested parameterizations remain fairly linear: on average about one order of magnitude smaller than the CRPs.

CONCLUSION

This paper examines how geometry of map projections can be extended to higher dimensions and then adopted for attitude parameterizations. It is shown that a set of three parameters based on product of the unit spin vector and any monotonically increasing function of the rotation angle that also passes through the origin can be used to represent attitude. These parameterizations collectively called the projected rotation parameters (PRPs) are shown to be kinematically lossless which allows them to be used in simple linear control laws with certain optimality characteristics. It is shown how many existing three-parameter representations of attitude fall within the general framework of the PRPs. Several new parameterizations inspired by standard map projections are introduced and their control performance examined numerically. Introducing various functions of the rotation angle for attitude parameterizations provides more opportunities for shaping attitude responses to the closed-loop control laws using these parameterizations.

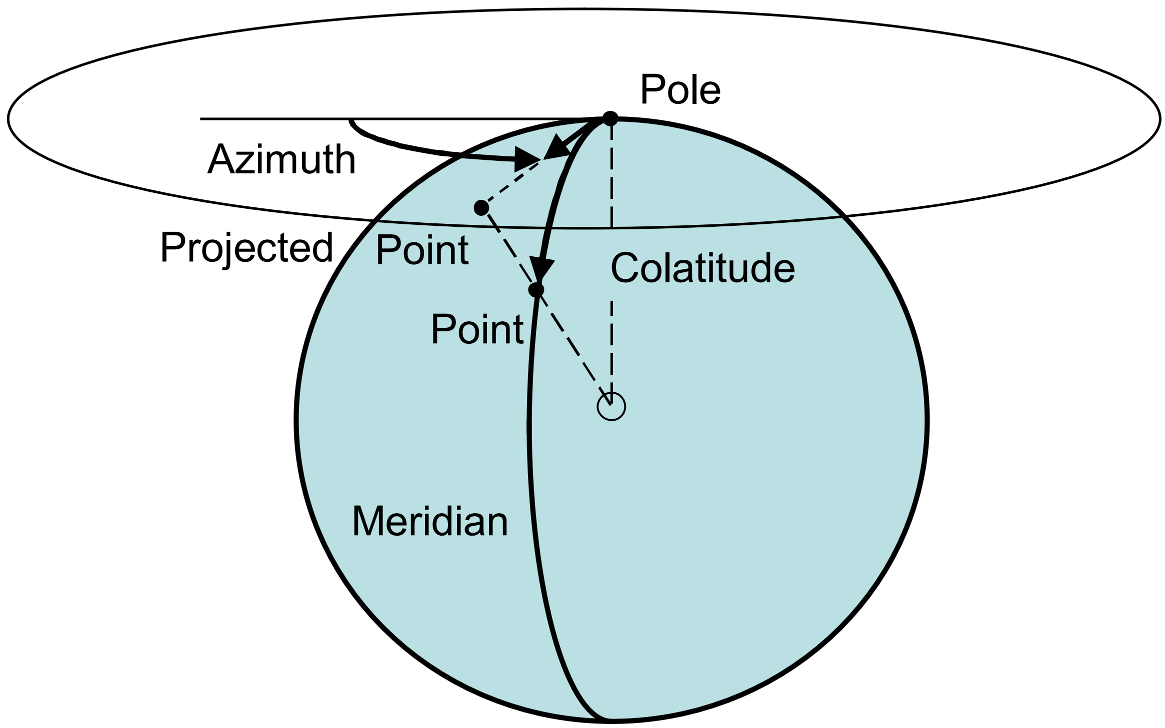

Figure 1. Example of Simple Azimuthal Map Projection Geometry: Gnomonic Projection.

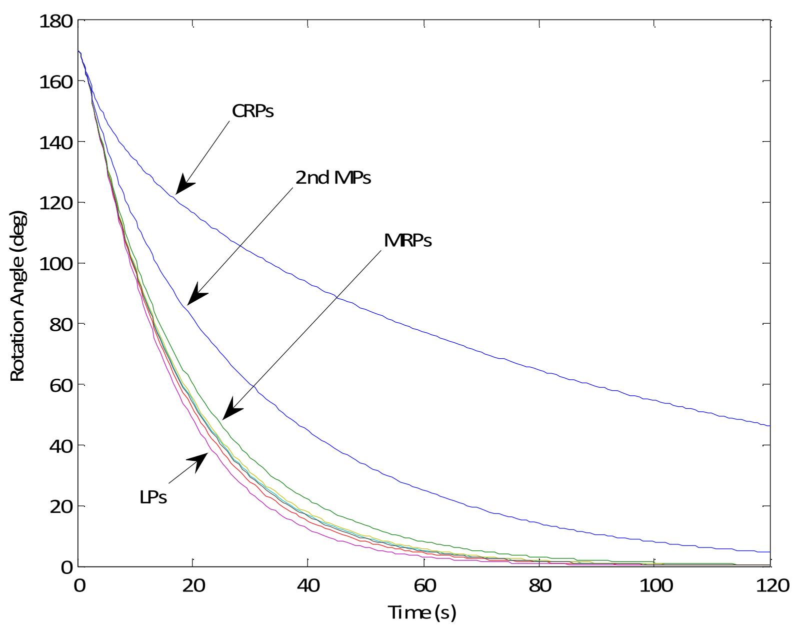

Figure 2. Rotation Angle for Various Types of PRPs.

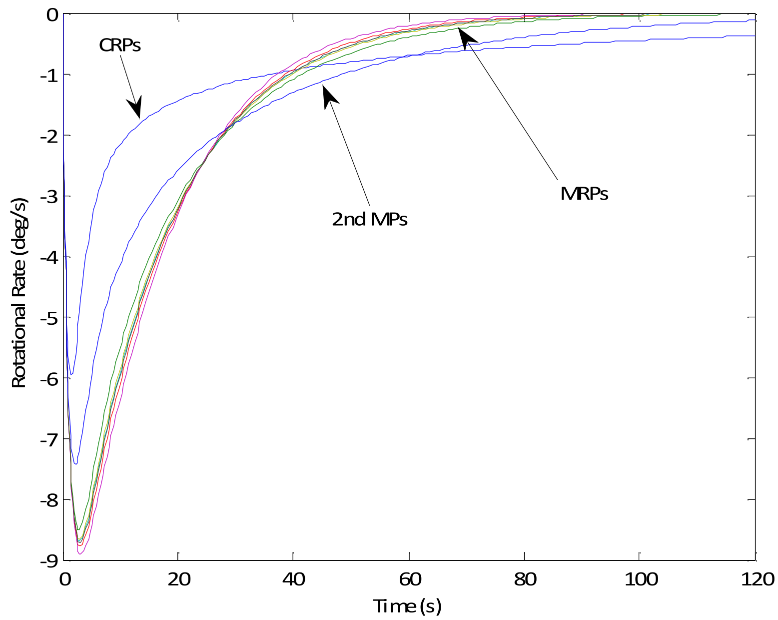

Figure 3. Rotational Rate for Various Types of PRPs.

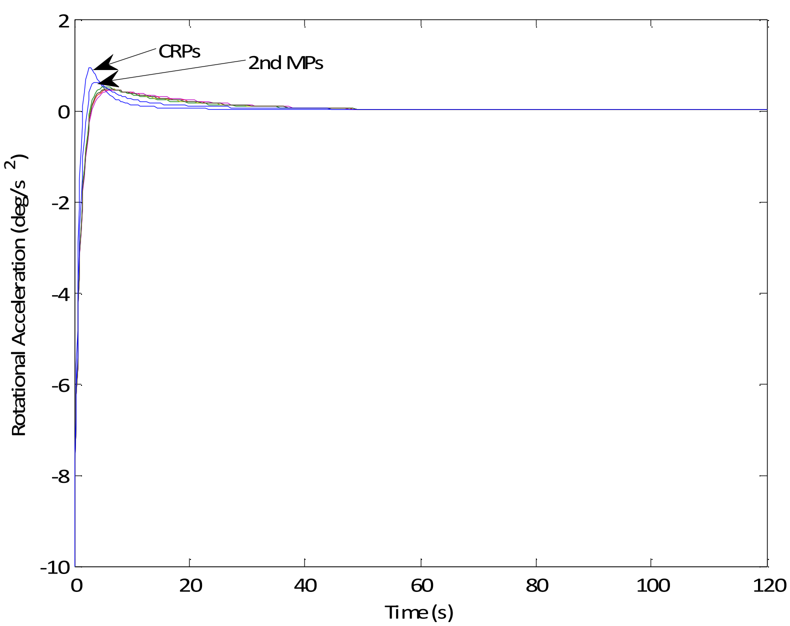

Figure 4. Rotational Acceleration for Various Types of PRPs.

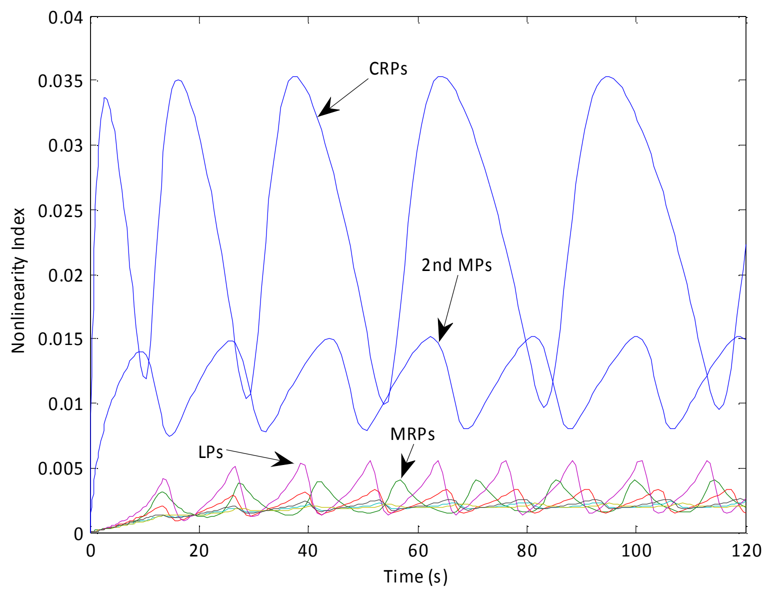

Figure 5. Nonlinearity Index for Various Types of PRPs.

APPENDIX: DETAILED DESCRIPTIONS OF VARIOUS TYPES OF PROJECTED ROTATION PARAMETERS

The following tables include closed-form expressions describing various relationships for the types of the PRPs introduced in this paper.

Table 3. Transformations between Various Types of PRPs.

Attitude Parameterization, $\mathbf{r}$	$\mathbf{r}(\boldsymbol{\rho}_m)$	$\boldsymbol{\rho}_m(\mathbf{r})$
EOPs, $\boldsymbol{\phi}$	$\boldsymbol{\phi} = 2\arctan\boldsymbol{\rho}$	$\boldsymbol{\rho} = \tan\dfrac{\boldsymbol{\phi}}{2}$
CRPs, $\boldsymbol{\rho}$	$\boldsymbol{\rho} = \boldsymbol{\rho}_1 = \boldsymbol{\sigma}\dfrac{2}{1-\sigma^2}$	$\boldsymbol{\rho}_1 = \boldsymbol{\rho}$
MRPs, $\boldsymbol{\sigma}$	$\boldsymbol{\sigma} = \boldsymbol{\rho}_2$	$\boldsymbol{\rho}_2 = \boldsymbol{\sigma}$
OPs, $\boldsymbol{\eta}$	$\boldsymbol{\eta} = \boldsymbol{\rho}\dfrac{1}{\sqrt{1+\rho^2}} = \boldsymbol{\sigma}\dfrac{2}{1+\sigma^2}$	$\boldsymbol{\rho} = \boldsymbol{\eta}\dfrac{1}{\sqrt{1-\eta^2}}$
LPs, $\boldsymbol{\lambda}$	$\boldsymbol{\lambda} = \boldsymbol{\sigma}\dfrac{1}{\sqrt{1+\sigma^2}}$	$\boldsymbol{\sigma} = \boldsymbol{\lambda}\dfrac{1}{\sqrt{1-\lambda^2}}$

BPs, $\boldsymbol{\beta}$	$\boldsymbol{\beta} = \boldsymbol{\sigma}\dfrac{1}{\sqrt[4]{1+\sigma^2}}$	$\boldsymbol{\sigma} = \boldsymbol{\beta}u$, $u = \sqrt{\dfrac{\sqrt{4+\beta^4}-\beta^2}{2}}$
NPPs, $\mathbf{p}_-$	$\mathbf{p}_- = \boldsymbol{\sigma}\dfrac{2}{1+w\sigma^2}, w = \dfrac{D-1}{D+1}$	$\boldsymbol{\sigma} = \mathbf{p}_- y_-$, $y_- = \dfrac{1-z_-}{wp_-^2}$, $z_- = \sqrt{1-wp_-^2}$
PPPs, $\mathbf{p}_+$	$\mathbf{p}_+ = \boldsymbol{\sigma}\dfrac{2}{1+\sigma^2/w}$	$\boldsymbol{\sigma} = \mathbf{p}_+ y_+$, $y_+ = \dfrac{1-z_+}{p_+^2/w}$, $z_+ = \sqrt{1-p_+^2/w}$
MPs, $\boldsymbol{\mu}_m$	$\boldsymbol{\mu}_m = 2\operatorname{arctanh}(\boldsymbol{\rho}_m)$	$\boldsymbol{\rho}_m = \tanh\dfrac{\boldsymbol{\mu}_m}{2}$

Table 4. Differential Transformations between Various Types of PRPs.

Attitude Parameterization, $\mathbf{r}$	$\mathbf{r}'(\boldsymbol{\rho}_m)$	$\boldsymbol{\rho}_m'(\mathbf{r})$
EOPs, $\boldsymbol{\phi}$	$\boldsymbol{\phi}'(\boldsymbol{\rho}) = \dfrac{\phi}{\tan\dfrac{\phi}{2}}\left[\mathbf{I} - \dfrac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{\phi^2}\left(1-\dfrac{\sin\phi}{\phi}\right)\right]$	$\boldsymbol{\rho}'(\boldsymbol{\phi}) = \dfrac{1}{\phi}\tan\dfrac{\phi}{2}\left[\mathbf{I} + \dfrac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{\phi^2}\left(\dfrac{\phi}{\sin\phi}-1\right)\right]$
CRPs, $\boldsymbol{\rho}$	$\boldsymbol{\rho}'(\boldsymbol{\rho}_1) = \mathbf{I}$	$\boldsymbol{\rho}_1'(\boldsymbol{\rho}) = \mathbf{I}$
MRPs, $\boldsymbol{\sigma}$	$\boldsymbol{\sigma}'(\boldsymbol{\rho}_2) = \mathbf{I}$	$\boldsymbol{\rho}_2'(\boldsymbol{\sigma}) = \mathbf{I}$
OPs, $\boldsymbol{\eta}$	$\boldsymbol{\eta}'(\boldsymbol{\rho}) = \sqrt{1-\eta^2}\left(\mathbf{I} - \boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}\right)$	$\boldsymbol{\rho}'(\boldsymbol{\eta}) = \dfrac{1}{\sqrt{1-\eta^2}}\left(\mathbf{I} + \dfrac{\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}}{1-\eta^2}\right)$
LPs, $\boldsymbol{\lambda}$	$\boldsymbol{\lambda}'(\boldsymbol{\sigma}) = \sqrt{1-\lambda^2}\left(\mathbf{I} - \boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}}\right)$	$\boldsymbol{\sigma}'(\boldsymbol{\lambda}) = \dfrac{1}{\sqrt{1-\lambda^2}}\left(\mathbf{I} + \dfrac{\boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}}}{1-\lambda^2}\right)$
BPs, $\boldsymbol{\beta}$	$\boldsymbol{\beta}'(\boldsymbol{\sigma}) = \dfrac{1}{u}\left(\mathbf{I} - \dfrac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{2u^2}\right)$	$\boldsymbol{\sigma}'(\boldsymbol{\beta}) = u\left(\mathbf{I} + \dfrac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{2u^2-\beta^2}\right)$
NPPs, $\mathbf{p}_-$	$\mathbf{p}_-'(\boldsymbol{\sigma}) = \dfrac{1}{y_-}\left[\mathbf{I} - \dfrac{(1-z_-)\mathbf{p}_-\mathbf{p}_-^{\mathrm{T}}}{p_-^2}\right]$	$\boldsymbol{\sigma}'(\mathbf{p}_-) = y_-\left[\mathbf{I} + \dfrac{(1-z_-)\mathbf{p}_-\mathbf{p}_-^{\mathrm{T}}}{z_- p_-^2}\right]$
PPPs, $\mathbf{p}_+$	$\mathbf{p}_+'(\boldsymbol{\sigma}) = \dfrac{1}{y_+}\left[\mathbf{I} - \dfrac{(1-z_+)\mathbf{p}_+\mathbf{p}_+^{\mathrm{T}}}{p_+^2}\right]$	$\boldsymbol{\sigma}'(\mathbf{p}_+) = y_+\left[\mathbf{I} + \dfrac{(1-z_+)\mathbf{p}_+\mathbf{p}_+^{\mathrm{T}}}{z_+ p_+^2}\right]$

MPs, $\boldsymbol{\mu}_m$	$\boldsymbol{\mu}'_m(\boldsymbol{\rho}_m) = \left[\mathbf{I} - \boldsymbol{\mu}_m\boldsymbol{\mu}_m^{\mathrm{T}} \frac{\mu_m \operatorname{csch}(\mu_m) - 1}{\mu_m^3 \operatorname{csch}(\mu_m)}\right] \times \mu_m \left(\tanh\frac{\mu_m}{2}\right)^{-1}$	$\boldsymbol{\rho}'_m(\boldsymbol{\mu}_m) = \left[\mathbf{I} + \boldsymbol{\mu}_m\boldsymbol{\mu}_m^{\mathrm{T}} \frac{\mu_m \operatorname{csch}(\mu_m) - 1}{\mu_m^2}\right] \times \frac{1}{\mu_m}\tanh\frac{\mu_m}{2}$

Table 3. Rotation Matrices for Various Types of PRPs.

Attitude Parameterization, $\mathbf{r}$	$\mathbf{C}(\mathbf{r})$
EOPs, $\boldsymbol{\phi}$	$\mathbf{C} = \mathbf{I} + \sin\phi[\![\hat{\mathbf{n}}]\!] + (1-\cos\phi)[\![\hat{\mathbf{n}}]\!]^2 = \left(\mathbf{I} + \left[\!\!\left[\tan\frac{\boldsymbol{\phi}}{2}\right]\!\!\right]\right)\left(\mathbf{I} - \left[\!\!\left[\tan\frac{\boldsymbol{\phi}}{2}\right]\!\!\right]\right)^{-1}$
CRPs, $\boldsymbol{\rho}$	$\mathbf{C} = \mathbf{I} + \frac{2}{1+\rho^2}[\![\boldsymbol{\rho}]\!] + \frac{2}{1+\rho^2}[\![\boldsymbol{\rho}]\!]^2 = (\mathbf{I} + [\![\boldsymbol{\rho}]\!])(\mathbf{I} - [\![\boldsymbol{\rho}]\!])^{-1}$
MRPs, $\boldsymbol{\sigma}$	$\mathbf{C} = \mathbf{I} + 4\frac{1-\sigma^2}{(1+\sigma^2)^2}[\![\boldsymbol{\sigma}]\!] + \frac{8}{(1+\sigma^2)^2}[\![\boldsymbol{\sigma}]\!]^2 = (\mathbf{I} + [\![\boldsymbol{\sigma}]\!])^2(\mathbf{I} - [\![\boldsymbol{\sigma}]\!])^{-2}$
OPs, $\boldsymbol{\eta}$	$\mathbf{C} = \mathbf{I} + 2\sqrt{1-\eta^2}[\![\boldsymbol{\eta}]\!] + 2[\![\boldsymbol{\eta}]\!]^2 = \left(\mathbf{I} + \frac{1}{\sqrt{1-\eta^2}}[\![\boldsymbol{\eta}]\!]\right)\left(\mathbf{I} - \frac{1}{\sqrt{1-\eta^2}}[\![\boldsymbol{\eta}]\!]\right)^{-1}$
LPs, $\boldsymbol{\lambda}$	$\mathbf{C} = \mathbf{I} + 4(1-2\lambda^2)\sqrt{1-\lambda^2}[\![\boldsymbol{\lambda}]\!] + 8(1-\lambda^2)[\![\boldsymbol{\lambda}]\!]^2$ $= \left(\mathbf{I} + \frac{1}{1\sqrt{1-\lambda^2}}[\![\boldsymbol{\lambda}]\!]\right)^2\left(\mathbf{I} - \frac{1}{\sqrt{1-\lambda^2}}[\![\boldsymbol{\lambda}]\!]\right)^{-2}$
BPs, $\boldsymbol{\beta}$	$\mathbf{C} = \mathbf{I} + 4(2u^7 - u^3)[\![\boldsymbol{\beta}]\!] + 8u^6[\![\boldsymbol{\beta}]\!]^2 = \left(\mathbf{I} + \frac{1}{u}[\![\boldsymbol{\beta}]\!]\right)^2\left(\mathbf{I} - \frac{1}{u}[\![\boldsymbol{\beta}]\!]\right)^{-2}$ $u = \sqrt{\frac{\sqrt{4+\beta^4} - \beta^2}{2}}$
NPPs, $\mathbf{p}_-$	$\mathbf{C} = \left(\mathbf{I} + \frac{1}{y_-}[\![\mathbf{p}_-]\!]\right)^2\left(\mathbf{I} - \frac{1}{y_-}[\![\mathbf{p}_-]\!]\right)^{-2}$
PPPs, $\mathbf{p}_+$	$\mathbf{C} = \left(\mathbf{I} + \frac{1}{y_+}[\![\mathbf{p}_+]\!]\right)^2\left(\mathbf{I} - \frac{1}{y_+}[\![\mathbf{p}_+]\!]\right)^{-2}$

MPs, $\boldsymbol{\mu}_m$	$\mathbf{C} = \left(\mathbf{I} + \left[\!\left[\tanh\dfrac{\boldsymbol{\mu}_m}{2}\right]\!\right]\right)^m \left(\mathbf{I} - \left[\!\left[\tanh\dfrac{\boldsymbol{\mu}_m}{2}\right]\!\right]\right)^{-m}$

Table 4. Direct Kinematics for Various Types of PRPs.

Attitude Parameterization	$\dot{\mathbf{r}} = \mathbf{G}(\mathbf{r})\boldsymbol{\omega}$
EOPs, $\boldsymbol{\phi}$	$\dot{\boldsymbol{\phi}} = \dfrac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \dfrac{\phi\sin\phi}{2(1-\cos\phi)}\boldsymbol{\omega} + \dfrac{1}{\phi^2}\left[1 - \dfrac{\phi(1+\cos\phi)}{2\sin\phi}\right](\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\omega})\boldsymbol{\phi}$
CRPs, $\boldsymbol{\rho}$	$\dot{\boldsymbol{\rho}} = \dfrac{1}{2}\left[\boldsymbol{\omega} + \boldsymbol{\rho}\times\boldsymbol{\omega} + \boldsymbol{\rho}(\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\omega})\right]$
MRPs, $\boldsymbol{\sigma}$	$\dot{\boldsymbol{\sigma}} = \dfrac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \dfrac{1}{4}\boldsymbol{\omega}\left[1-\sigma^2\right] + \dfrac{1}{2}(\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\sigma})\boldsymbol{\sigma}$
OPs, $\boldsymbol{\eta}$	$\dot{\boldsymbol{\eta}} = \dfrac{1}{2}\boldsymbol{\eta}\times\boldsymbol{\omega} + \dfrac{1}{2}\boldsymbol{\omega}\sqrt{1-\eta^2}$
LPs, $\boldsymbol{\lambda}$	$\dot{\boldsymbol{\lambda}} = \dfrac{1}{2}\boldsymbol{\lambda}\times\boldsymbol{\omega} + \dfrac{1}{4}\boldsymbol{\omega}\dfrac{1-2\lambda^2}{\sqrt{1-\lambda^2}} + \dfrac{1}{4}(\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\lambda})\boldsymbol{\lambda}\dfrac{1}{\sqrt{1-\lambda^2}}$
BPs, $\boldsymbol{\beta}$	$\dot{\boldsymbol{\beta}} = \dfrac{1}{2}\boldsymbol{\beta}\times\boldsymbol{\omega} + \dfrac{1}{4}\boldsymbol{\omega}\left[u - \beta^2\dfrac{1}{u}\right] + \dfrac{3}{8}(\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\beta})\boldsymbol{\beta}\dfrac{1}{u}$
NPPs, $\mathbf{p}_-$	$\dot{\mathbf{p}}_- = \dfrac{1}{2}\mathbf{p}_-\times\boldsymbol{\omega} + \dfrac{1-p_-^2y_-^2}{4y_-}\boldsymbol{\omega} + \dfrac{1-w}{4}(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{p}_-)\mathbf{p}_-$
PPPs, $\mathbf{p}_+$	$\dot{\mathbf{p}}_+ = \dfrac{1}{2}\mathbf{p}_+\times\boldsymbol{\omega} + \dfrac{1-p_+^2y_+^2}{4y_+}\boldsymbol{\omega} + \dfrac{1-1/w}{4}(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{p}_+)\mathbf{p}_+$
MPs, $\boldsymbol{\mu}_2$	$\dot{\boldsymbol{\mu}}_2 = \dfrac{1}{2}\boldsymbol{\mu}_2\times\boldsymbol{\omega} + \dfrac{1}{2}\boldsymbol{\omega}\mu_2\operatorname{csch}(\mu_2) + \dfrac{1}{2}(\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\mu}_2)\boldsymbol{\mu}_2\left[\dfrac{\cosh(\mu_2) - \mu_2\operatorname{csch}(\mu_2)}{\mu_2^2}\right]$

Table 5. Inverse Kinematics for Various Types of PRPs.

Attitude Parameteri-zation	$\boldsymbol{\omega} = \mathbf{H}(\mathbf{r})\dot{\mathbf{r}}$
EOPs, $\boldsymbol{\phi}$	$\boldsymbol{\omega} = \dot{\boldsymbol{\phi}}\dfrac{\sin\phi}{\phi} - \dfrac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} - \dfrac{1}{\phi^2}\left[\dfrac{\sin\phi}{\phi} - 1\right](\boldsymbol{\phi}^{\mathrm{T}}\dot{\boldsymbol{\phi}})\boldsymbol{\phi}$

CRPs, $\boldsymbol{\rho}$	$\boldsymbol{\omega} = \dfrac{2}{1+\rho^2}\left[\dot{\boldsymbol{\rho}} - \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}\right]$
MRPs, $\boldsymbol{\sigma}$	$\boldsymbol{\omega} = \dfrac{4}{\left[1+\sigma^2\right]^2}\left[\dot{\boldsymbol{\sigma}}\left(1-\sigma^2\right) - 2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} + 2\left(\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}\right)\boldsymbol{\sigma}\right]$
OPs, $\boldsymbol{\eta}$	$\boldsymbol{\omega} = 2\left[\dot{\boldsymbol{\eta}}\sqrt{1-\eta^2} - \boldsymbol{\eta}\times\dot{\boldsymbol{\eta}} + \dfrac{1}{\sqrt{1-\eta^2}}\left(\boldsymbol{\eta}^{\mathrm{T}}\dot{\boldsymbol{\eta}}\right)\boldsymbol{\eta}\right]$
LPs, $\boldsymbol{\lambda}$	$\boldsymbol{\omega} = 4\left[\dot{\boldsymbol{\lambda}}\left(1-2\lambda^2\right)\sqrt{1-\lambda^2} - 2\left(1-\lambda^2\right)\boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}} + \dfrac{3-2\lambda^2}{\sqrt{1-\lambda^2}}\left(\boldsymbol{\lambda}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}\right)\boldsymbol{\lambda}\right]$
BPs, $\boldsymbol{\beta}$	$\boldsymbol{\omega} = 4\left[\dot{\boldsymbol{\beta}}u^3\left(2u^4-1\right) - 2u^6\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} + \dfrac{u^5\left(2u^4+3\right)}{1+u^4}\left(\boldsymbol{\beta}^{\mathrm{T}}\dot{\boldsymbol{\beta}}\right)\boldsymbol{\beta}\right]$
NPPs, $\mathbf{p}_{-}$	$\boldsymbol{\omega} = \dfrac{4y_{-}}{\left[1+p_{-}^2y_{-}^2\right]^2}\left[\dot{\mathbf{p}}_{-}\left(1-p_{-}^2y_{-}^2\right) - 2y_{-}\mathbf{p}_{-}\times\dot{\mathbf{p}}_{-} + \dfrac{1+w}{z\left(1+z_{-}\right)}\left(\mathbf{p}_{-}^{\mathrm{T}}\dot{\mathbf{p}}_{-}\right)\mathbf{p}_{-}\right]$
PPPs, $\mathbf{p}_{+}$	$\boldsymbol{\omega} = \dfrac{4y_{+}}{\left[1+p_{+}^2y_{+}^2\right]^2}\left[\dot{\mathbf{p}}_{-}\left(1-p_{+}^2y_{+}^2\right) - 2y_{+}\mathbf{p}_{+}\times\dot{\mathbf{p}}_{+} + \dfrac{1+1/w}{z\left(1+z_{+}\right)}\left(\mathbf{p}_{+}^{\mathrm{T}}\dot{\mathbf{p}}_{+}\right)\mathbf{p}_{+}\right]$
MPs, $\boldsymbol{\mu}_2$	$\boldsymbol{\omega} = 2\dfrac{\tanh(\mu_2)}{\mu_2^2}\left[\begin{array}{l}\dot{\boldsymbol{\mu}}_2\mu_2\mathrm{sech}(\mu_2) - \tanh(\mu_2)\boldsymbol{\mu}_2\times\dot{\boldsymbol{\mu}}_2 \\ +\boldsymbol{\mu}_2\left(\boldsymbol{\mu}_2^{\mathrm{T}}\dot{\boldsymbol{\mu}}_2\right)\left[\dfrac{\mu_2\mathrm{csch}(\mu_2)-1}{\mu_2}\right]\end{array}\right]$

Table 6. Storage Functions for Various Types of PRPs.

Attitude Parameterization, $\mathbf{r}$	$V(\mathbf{r}) = \int_0^{\phi} r(\xi)d\xi$
EOPs, $\boldsymbol{\phi}$	$V(\boldsymbol{\phi}) = \dfrac{1}{2}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}$
CRPs, $\boldsymbol{\rho}$	$V(\boldsymbol{\rho}) = \ln\left(1+\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\rho}\right)$
MRPs, $\boldsymbol{\sigma}$	$V(\boldsymbol{\sigma}) = 2\ln\left(1+\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}\right)$
OPs, $\boldsymbol{\eta}$	$V(\boldsymbol{\eta}) = 2\left(1-\cos\dfrac{\phi}{2}\right)$

LPs, $\boldsymbol{\lambda}$	$V(\boldsymbol{\lambda}) = 4\left(1-\cos\frac{\phi}{4}\right)$
BPs, $\boldsymbol{\beta}$	$V(\boldsymbol{\beta}) = 8(1-u)$
NPPs, $\mathbf{p}_-$	$V(\mathbf{p}_-) = \ln\left[\left(D+\cos\frac{\phi}{2}\right)^{-2(D+1)}\right] + \ln\left[(D+1)^{2(D+1)}\right]$
PPPs, $\mathbf{p}_+$	$V(\mathbf{p}_+) = \ln\left[\left(D-\cos\frac{\phi}{2}\right)^{2(D-1)}\right] - \ln\left[(D-1)^{2(D-1)}\right]$
MPs, $\boldsymbol{\mu}_m$	$V(\boldsymbol{\mu}_m) = 2\phi\mu_m + 2mi\left(\frac{\phi}{m}\arctan(e^{i\frac{\phi}{m}}) + \chi_2(ie^{i\frac{\phi}{m}}) - \chi_2(i)\right)^*$

REFERENCES

[1] P. Tsiotras and J. M. Longuski, "A New Parameterization of the Attitude Kinematics," *Journal of the Astronautical Sciences*, Vol. 43, No. 3, 1995, pp. 243-262.

[2] H. Schaub and J. L. Junkins, "Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters," *Journal of the Astronautical Sciences*, Vol. 44, No. 1, 1996, pp. 13-15.

[3] J. Mullen and H. Schaub, "Hypersphere Stereographic Orientation Parameters," Paper AAS 10-228, *20th AAS/AIAA Space Flight Mechanics Meeting*, San Diego, CA, Feb. 14-18, 2010.

[4] L. M. Bugayevskiy and J. P. Snyder, *Map Projections: A Reference Manual*. Taylor and Francis Inc., Philadelphia, 2000, p. 101.

[5] M. D. Shuster, "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439-517.

[6] P. Tsiotras, "Further Passivity Results for the Attitude Control Problem," *IEEE Transactions on Automatic Control*, Vol. 43, No. 11, 1998, pp. 1597-1600.

[7] P. Tsiotras, "New Control Laws for the Attitude Stabilization of Rigid Bodies," *13th IFAC Symposium on Automatic Control in Aerospace*, Palo Alto, CA Sept. 12-16, 1994, pp. 316-321.

[8] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey, 3rd ed., 2002.

[9] P. Tsiotras, "Stabilization and Optimality Results for the Attitude Control Problem," *Journal of Guidance, Control and Dynamics*, Vol. 19, No. 4, 1996, pp. 772-779.

[10] P. Tsiotras, J. L. Junkins and H. Schaub, "Higher Order Cayley Transforms with Applications to Attitude Representations," *Journal of Guidance, Control and Dynamics*, Vol. 20, No. 3, 1997, pp. 528-534.

[11] P. Tsiotras, "Asymptotic Properties of Higher Order Cayley Transforms," Paper AIAA 98-4387, *AIAA/AAS Astrodynamics Specialist Conference*, Boston, MA, Aug. 10-12, 1998.

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[*] Here χ_2 denotes the 2nd order Legendre's Chi-function.