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SOME DIRECTIONS IN SPIN-AXIS ATTITUDE

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Du spinnst doch! / Du hast doch'n Knall!
Du spinnst doch! / Du spinnst doch total!‡

Die Prinzen, 1994

Four methods of spin-axis estimation are presented and examined in a manner analogous to the study of three-axis attitude estimation with a particular emphasis on the treatment of constraint. Accuracy and efficiency of the methods are compared both numerically and analytically including a thorough covariance analysis.

INTRODUCTION

Spin-axis attitude estimation is in some respects analogous to three-axis attitude estimation. For example, both use known reference vectors along with measurements from spacecraft-based sensors to estimate spacecraft attitude. Of course, there are also fundamental differences. Three-axis attitude estimation seeks a complete attitude solution, which belongs to the group of rotations, while spin-axis attitude estimation seeks only a spin-axis direction, which certainly does not belong to a group. In addition, the typical measurements for three-axis attitude estimation are directions, whereas those for spin-axis estimation are usually angles (or, equivalently, cosines). Nevertheless, there are interesting parallels between the two estimation domains, one of which, the presence of a constraint, is the subject of this study.

This work is an extension of Ref. 1, which, among other things, includes a thorough discussion of the measurements and models employed in spin-axis estimation. The models are linear in the spin-axis unit vector, which is very helpful in developing estimation methods within the framework of maximum-likelihood estimation, an approach that also works well for three-axis attitude estimation. However, the proper treatment of the spin-axis norm constraint is of concern. We will reexamine different spin-axis

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‡ You're like really so spinning! / You're like really so whacked! / You're like really so spinning! / You're really spinning to the max! Before becoming a leading European rock group, the “Princes” were all members of the choir of the Thomaskirche in Leipzig, whose most illustrious choirmaster was Johann Sebastian Bach.

attitude estimation methods and provide statistically sound recommendations based on their accuracy and efficiency.

MEASUREMENT MODELS AND THE COST FUNCTION

We begin by defining an effective measurement vector which is linear in the spin-axis unit vector $\hat{\mathbf{n}}$ and satisfies

$$\mathbf{Z}_k = H_k \hat{\mathbf{n}} + \mathbf{v}_k, \quad k = 1, \ldots, N \tag{1}$$

with white measurement noise

$$\mathbf{v}_k \sim N(\mathbf{0}, R_K), \quad k = 1, \ldots, N \tag{2}$$

Here the rows of sensitivity matrix H_k are known reference vectors [1], and the effective measurement vector $\mathbf{Z}_k$ consists of cosines of angles between these vectors and the spin-axis direction. The subscript k indicates the frame number. There are, of course, many choices for the effective measurements, depending on the sensor suite and the choice of dihedral angles.

We construct the data-dependent part of the negative-log-likelihood function [2-4] using the above measurement model

$$J(\hat{\mathbf{n}}) = \frac{1}{2} \sum_{k=1}^{N} \left(\mathbf{Z}_k - H_k \hat{\mathbf{n}} \right)^{\mathrm{T}} R_k^{-1} \left(\mathbf{Z}_k - H_k \hat{\mathbf{n}} \right) \tag{3}$$

where T denotes the matrix transpose. The maximum-likelihood estimate [3] is simply

$$\hat{\mathbf{n}}^* \equiv \arg \min_{|\hat{\mathbf{n}}|=1} J(\hat{\mathbf{n}}) \tag{4}$$

Any method that minimizes $J(\hat{\mathbf{n}})$ directly in terms of $\hat{\mathbf{n}}$ must take account of the norm constraint

$$\hat{\mathbf{n}}^{\mathrm{T}} \hat{\mathbf{n}} = 1 \tag{5}$$

The cost function $J(\hat{\mathbf{n}})$ can be expanded as

$$J(\hat{\mathbf{n}}) = \mathsf{J} + \mathbf{G}^{\mathrm{T}} \hat{\mathbf{n}} + \frac{1}{2} \hat{\mathbf{n}}^{\mathrm{T}} \mathsf{F} \hat{\mathbf{n}} \tag{6}$$

with

$$\mathsf{J} \equiv \frac{1}{2} \sum_{k=1}^{N} \mathbf{Z}_k^{\mathrm{T}} R_k^{-1} \mathbf{Z}_k \tag{7a}$$

$$\mathbf{G} \equiv -\sum_{k=1}^{N} H_k^{\mathrm{T}} R_k^{-1} \mathbf{Z}_k \tag{7b}$$

$$\mathsf{F} \equiv \sum_{k=1}^{N} H_k^{\mathrm{T}} R_k^{-1} H_k \tag{7c}$$

The scalar J, the column matrix $\mathbf{G}$, and the symmetric matrix F are introduced for computational convenience as they encapsulate all the individual measurements. The matrix F is positive semidefinite, and positive definite if the measurements are not all coplanar. We focus principally on the non-singular case in this study.

There are two approaches to finding the minimizing value of $\hat{\mathbf{n}}$. One approach uses the original three-dimensional variable $\hat{\mathbf{n}}$, but requires additional operations to satisfy the constraint of equation (5) at each step; the other approach satisfies the constraint automatically by changing the argument of the cost function to a two-dimensional variable. We consider both approaches in the present work.

LAGRANGE-MULTIPLIER METHOD

We consider Lagrange's method of multipliers, for which we write the augmented cost function [1]

$$J'(\hat{\mathbf{n}}) = J(\hat{\mathbf{n}}) + \frac{1}{2}\lambda\left(\hat{\mathbf{n}}^{\mathrm{T}}\hat{\mathbf{n}} - 1\right) \tag{8}$$

with λ a yet undetermined constant, and minimize $J'(\hat{\mathbf{n}})$ without constraint leading to

$$\left(\frac{\partial J'}{\partial \hat{\mathbf{n}}}(\hat{\mathbf{n}}^*)\right)^{\mathrm{T}} = \mathbf{G} + \left(\mathsf{F} + \lambda I_{3\times3}\right)\hat{\mathbf{n}}^* = \mathbf{0} \tag{9}$$

This yields both the explicitly constrained estimate

$$\hat{\mathbf{n}}^* = -\left(\mathsf{F} + \lambda I_{3\times3}\right)^{-1}\mathbf{G} \tag{10}$$

And, from the constraint, the equation for the Lagrange multiplier

$$\left(\hat{\mathbf{n}}^*\right)^{\mathrm{T}}\hat{\mathbf{n}}^* = \mathbf{G}^{\mathrm{T}}\left(\mathsf{F} + \lambda I_{3\times3}\right)^{-2}\mathbf{G} = 1 \tag{11}$$

The latter can be used to solve for λ (and $\hat{\mathbf{n}}^*$) by the Newton-Raphson method

$$\lambda_0 = 0 \tag{12a}$$

$$D_{i-1} = \left(\mathsf{F} + \lambda_{i-1} I_{3\times3}\right)^{-1} \tag{12b}$$

$$\mathbf{n}_{i-1} = -D_{i-1}\mathbf{G} \tag{12c}$$

$$\lambda_i = \lambda_{i-1} - \frac{1}{2}\frac{\left(1-\mathbf{n}_{i-1}^{\mathrm{T}}\mathbf{n}_{i-1}\right)}{\mathbf{n}_{i-1}^{\mathrm{T}}D_{i-1}\mathbf{n}_{i-1}} \tag{12d}$$

and

$$\lambda = \lim_{i\to\infty}\lambda_i\,, \qquad \hat{\mathbf{n}}^* = \lim_{i\to\infty}\mathbf{n}_i \tag{12ef}$$

Based on equation (10), the estimation error evaluated about the noise-free (or true) solution is given to first order in the measurement noise by

$$\Delta\hat{\mathbf{n}}^* = -\mathsf{F}^{-1}\left(\Delta\mathbf{G}-\left(\Delta\lambda\right)\hat{\mathbf{n}}^{\mathrm{true}}\right) \tag{13a}$$

with

$$\Delta\mathbf{G} = -\sum_{k=1}^{N}H_k^{\mathrm{T}}R_k^{-1}\mathbf{v}_k \tag{13b}$$

The values of the spin-axis vector and the Lagrange multiplier for the true values of the measurements are

$$\hat{\mathbf{n}}^{\mathrm{true}} = -\mathsf{F}^{-1}\mathbf{G}^{\mathrm{true}}, \qquad \lambda^{\mathrm{true}} = 0 \tag{14ab}$$

Note that

$$E\left\{\Delta\mathbf{G}\right\} = \mathbf{0} \qquad \text{and} \qquad E\left\{\Delta\mathbf{G}\Delta\mathbf{G}^{\mathrm{T}}\right\} = \mathsf{F} \tag{15ab}$$

where $E\{\,\cdot\,\}$ denotes the expectation. The first-order error in λ can be obtained from the constraint equation (Eq.(11)) with some effort as

$$\Delta\lambda = \left(\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}}\mathsf{F}^{-1}\hat{\mathbf{n}}^{\mathrm{true}}\right)^{-1}\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}}\mathsf{F}^{-1}\Delta\mathbf{G} \tag{16}$$

from which it immediately follows that

$$E\left\{\Delta\lambda\right\} = 0\,, \qquad E\left\{\left(\Delta\lambda\right)^2\right\} = \left(\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}}\mathsf{F}^{-1}\hat{\mathbf{n}}^{\mathrm{true}}\right)^{-1} \tag{17ab}$$

and

$$E\left\{\left(\Delta\lambda\right)\Delta\mathbf{G}\right\} = \left(\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}}\mathsf{F}^{-1}\hat{\mathbf{n}}^{\mathrm{true}}\right)^{-1}\hat{\mathbf{n}}^{\mathrm{true}} \tag{17c}$$

We can see from equation (17b) that the root-mean-square (RMS) value of λ and, consequently, of $\left|\mathbf{G}+\mathsf{F}\hat{\mathbf{n}}^*\right|$, will be of order $\sqrt{N/\sigma^2}$, i.e. generally much smaller than N/σ^2, the order of F.

The calculation of the spin-axis attitude covariance matrix is now straightforward with the result

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = \Lambda\mathsf{F}^{-1}\Lambda^{\mathrm{T}} \tag{18}$$

where

$$\Lambda = I_{3\times3} - \left(\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}}\mathsf{F}^{-1}\hat{\mathbf{n}}^{\mathrm{true}}\right)^{-1}\mathsf{F}^{-1}\hat{\mathbf{n}}^{\mathrm{true}}\left(\hat{\mathbf{n}}^{\mathrm{true}}\right)^{\mathrm{T}} \tag{19}$$

Note that this covariance matrix is singular and that, as expected, the true spin-axis vector lies in its null-space

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\hat{\mathbf{n}}^{\text{true}} = \mathbf{0} \tag{20}$$

UNCONSTRAINED BRUTE-FORCE METHOD

The second estimation method that deals directly with $\hat{\mathbf{n}}$ initially finds an unconstrained estimate $\mathbf{n}_{\text{uc}}^*$ that minimizes $J(\hat{\mathbf{n}})$ without constraint. This leads immediately to

$$\left(\frac{\partial J}{\partial \hat{\mathbf{n}}}(\mathbf{n}_{\text{uc}}^*)\right)^{\text{T}} = \mathsf{G} + \mathsf{F}\mathbf{n}_{\text{uc}}^* = \mathbf{0} \tag{21}$$

or

$$\mathbf{n}_{\text{uc}}^* = -\mathsf{F}^{-1}\mathsf{G} \tag{22}$$

The constrained estimate of equation (10) can now be rewritten in terms of this unconstrained estimate as

$$\hat{\mathbf{n}}^* = \left(I_{3\times 3} + \lambda \mathsf{F}^{-1}\right)^{-1}\mathbf{n}_{\text{uc}}^* \tag{23}$$

While, in general, $\mathbf{n}_{\text{uc}}^*$ does not have unit norm and, thus, cannot be a direction, in the absence of measurement noise it must coincide with the true spin-axis $\hat{\mathbf{n}}^{\text{true}}$. From this we infer that the two must be close in value and that $\mathbf{n}_{\text{uc}}^*$ must also be close to $\hat{\mathbf{n}}^*$. We can define the "estimate error"

$$\Delta\mathbf{n}_{\text{uc}}^* \equiv \mathbf{n}_{\text{uc}}^* - \hat{\mathbf{n}}^{\text{true}} = \mathsf{F}^{-1}\Delta\mathsf{G} \tag{24}$$

and the corresponding covariance matrix must satisfy

$$P_{\mathbf{nn}}^{\text{uc}} \equiv E\left\{\Delta\mathbf{n}_{\text{uc}}^*\left(\Delta\mathbf{n}_{\text{uc}}^*\right)^{\text{T}}\right\} = \mathsf{F}^{-1} \tag{25}$$

The brute-force normalization of $\mathbf{n}_{\text{uc}}^*$

$$\hat{\mathbf{n}}_{\text{uc}}^* \equiv \frac{\mathbf{n}_{\text{uc}}^*}{\left|\mathbf{n}_{\text{uc}}^*\right|} \tag{26}$$

results in

$$\Delta\hat{\mathbf{n}}_{\text{uc}}^* = \left(I_{3\times 3} - \hat{\mathbf{n}}^{\text{true}}\left(\hat{\mathbf{n}}^{\text{true}}\right)^{\text{T}}\right)\Delta\mathbf{n}_{\text{uc}}^* \tag{27}$$

and the covariance matrix for the brute force estimate is

$$P_{\hat{\mathbf{n}}_{\text{uc}}\hat{\mathbf{n}}_{\text{uc}}} = \Lambda_{\text{uc}}\mathsf{F}^{-1}\Lambda_{\text{uc}}^{\text{T}} \tag{28}$$

with

$$\Lambda_{\text{uc}} = I_{3\times3} - \hat{\mathbf{n}}^{\text{true}} \left(\hat{\mathbf{n}}^{\text{true}}\right)^{\text{T}} \tag{29}$$

analogous to equation (18). Note, however, that the two covariance matrices will be very different if F shows strong correlations, in which case the unconstrained estimate will be a poor approximation. In any event, the computation of an explicitly constrained estimate is so simple, that the brute-force method has little practical value. At best, it provides only an approximate estimate.

INCREMENTAL-VECTOR METHOD

The incremental-vector method defines a (proper) orthonormal triad of vectors, $\{\hat{\mathbf{n}}_i, \hat{\mathbf{a}}_i, \hat{\mathbf{b}}_i\}$, where the last two vectors lie in the plane, the tangent plane perpendicular to the ith spin-axis estimate $\hat{\mathbf{n}}_i$. Using this triad, we may write

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i + O(|\boldsymbol{\varepsilon}_i|^2) \tag{30}$$

where $\boldsymbol{\varepsilon}_i$, the 2×1 column matrix of incremental variables, and the 3×2 matrix C_i are given by

$$\boldsymbol{\varepsilon}_i \equiv \begin{bmatrix} \varepsilon_{a,i} & \varepsilon_{b,i} \end{bmatrix}^{\text{T}} \quad \text{and} \quad C_i \equiv \begin{bmatrix} \hat{\mathbf{a}}_{i-1} & \hat{\mathbf{b}}_{i-1} \end{bmatrix} \tag{31ab}$$

We may rewrite the cost function $J(\hat{\mathbf{n}})$ in terms of the incremental variables

$$J(\hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i) = \mathsf{J} + \mathbf{G}^{\text{T}} \left(\hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i\right) + \frac{1}{2}\left(\hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i\right)^{\text{T}} \mathsf{F} \left(\hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i\right) \tag{32}$$

and minimize it to find

$$\boldsymbol{\varepsilon}_i^* = -\left(C_i^{\text{T}} \mathsf{F} C_i\right)^{-1} C_i^{\text{T}} \left(\mathbf{G} + \mathsf{F}\hat{\mathbf{n}}_{i-1}\right) \tag{33}$$

and

$$\hat{\mathbf{n}}_i = \hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i^* + O(|\boldsymbol{\varepsilon}_i|^2) \tag{34}$$

In practice, we replace the last equation with

$$\mathbf{n}_i = \hat{\mathbf{n}}_{i-1} + C_i \boldsymbol{\varepsilon}_i^* \quad \text{and} \quad \hat{\mathbf{n}}_i = \mathbf{n}_i / |\mathbf{n}_i| \tag{35ab}$$

So that we have a unit vector at every step of the iteration. When the iterations are initialized with

$$\hat{\mathbf{n}}_0 \equiv \hat{\mathbf{n}}_{\text{uc}}^* \tag{36}$$

and we must have

$$\lim_{i\to\infty} \boldsymbol{\varepsilon}_i^* = \mathbf{0} \quad \text{and} \quad \lim_{i\to\infty} \hat{\mathbf{n}}_i = \hat{\mathbf{n}}^* \tag{37ab}$$

It is important to emphasize the distinction between the estimate error $\tilde{\boldsymbol{\varepsilon}}$ and the incremental vector $\boldsymbol{\varepsilon}_i$. The former is based on the true triad $\{\hat{\mathbf{n}}^{\text{true}}, \hat{\mathbf{a}}^{\text{true}}, \hat{\mathbf{b}}^{\text{true}}\}$, whereas the latter is based on the triad

$\left\{\hat{\mathbf{n}}_i, \hat{\mathbf{a}}_i, \hat{\mathbf{b}}_i\right\}$ defined by the ith spin-axis estimate. The computation of the Fisher information matrix is straightforward now that the two components of $\tilde{\boldsymbol{\varepsilon}}$ are independent

$$F_{\tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}}} = C^{\mathrm{T}} \mathsf{F} C \tag{38}$$

with $C \equiv \left[\hat{\mathbf{a}}^{\text{true}} \quad \hat{\mathbf{b}}^{\text{true}}\right]$. The estimate-error covariance matrix follows immediately as

$$P_{\tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}}} = F_{\tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}}}^{-1} \tag{39}$$

provided that the Fisher information matrix is invertible (i.e., the spin axis attitude is observable). The first-order relationship between the estimate error and the spin-axis attitude error is given by

$$\Delta\hat{\mathbf{n}}^* = C\tilde{\boldsymbol{\varepsilon}} \tag{40}$$

from which we obtain the spin-axis error covariance

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = C\left(C^{\mathrm{T}} \mathsf{F} C\right)^{-1} C^{\mathrm{T}} \tag{41}$$

or

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = C\left(C^{\mathrm{T}}\left(P_{\mathbf{nn}}^{\text{uc}}\right)^{-1} C\right)^{-1} C^{\mathrm{T}} \tag{42}$$

INCREMENTAL-ANGLE METHOD

The incremental-angle method uses two spherical angles, the polar angle θ_1 and the azimuthal angle θ_2, to parameterize the spin-axis vector

$$\hat{\mathbf{n}}(\boldsymbol{\theta}) = \begin{bmatrix} \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ \cos\theta_1 \end{bmatrix} \tag{43}$$

and the cost function

$$J(\boldsymbol{\theta}) = J(\hat{\mathbf{n}}(\boldsymbol{\theta})) \tag{44}$$

with $\boldsymbol{\theta} = \left[\theta_1 \quad \theta_2\right]^{\mathrm{T}}$. Then the gradient vector and Hessian matrix are given by

$$\left(\frac{\partial J}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta})\right)^{\mathrm{T}} = M^{\mathrm{T}}(\boldsymbol{\theta})\left(\mathbf{G} + \mathsf{F}\hat{\mathbf{n}}(\boldsymbol{\theta})\right) \tag{45}$$

$$\frac{\partial^2 J}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\mathrm{T}}}(\boldsymbol{\theta}) = M^{\mathrm{T}}(\boldsymbol{\theta}) \mathsf{F} M(\boldsymbol{\theta}) + \sum_{j=1}^{3}\left(\mathbf{G} + \mathsf{F}\hat{\mathbf{n}}(\boldsymbol{\theta})\right)_j \frac{\partial^2 \hat{n}_j}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\mathrm{T}}}(\boldsymbol{\theta}) \tag{46}$$

with

$$M(\boldsymbol{\theta}) \equiv \frac{\partial \hat{\mathbf{n}}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) = \begin{bmatrix} \cos\theta_1 \cos\theta_2 & \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \\ -\sin\theta_1 \sin\theta_2 & \sin\theta_1 \cos\theta_2 & 0 \end{bmatrix}^{\mathrm{T}} \tag{47}$$

The second term in equation (46) will nearly vanish when we evaluate the expectation to compute the Fisher information matrix. Therefore, we are led to good approximation to

$$F_{\boldsymbol{\theta\theta}} = M^{\mathrm{T}}(\boldsymbol{\theta}^{\mathrm{true}})\,\mathsf{F}\,M(\boldsymbol{\theta}^{\mathrm{true}}) \tag{48}$$

and, in a manner entirely analogous to equations (39, 41, 42), we obtain

$$P_{\boldsymbol{\theta\theta}} = F_{\boldsymbol{\theta\theta}}^{-1} \tag{49}$$

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = M(\boldsymbol{\theta}^{\mathrm{true}})\left(M^{\mathrm{T}}(\boldsymbol{\theta}^{\mathrm{true}})\,\mathsf{F}\,M(\boldsymbol{\theta}^{\mathrm{true}})\right)^{-1} M^{\mathrm{T}}(\boldsymbol{\theta}^{\mathrm{true}}) \tag{50}$$

or

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = M(\boldsymbol{\theta}^{\mathrm{true}})\left(M^{\mathrm{T}}(\boldsymbol{\theta}^{\mathrm{true}})\left(P_{\mathbf{nn}}^{\mathrm{uc}}\right)^{-1} M(\boldsymbol{\theta}^{\mathrm{true}})\right)^{-1} M^{\mathrm{T}}(\boldsymbol{\theta}^{\mathrm{true}}) \tag{51}$$

The Gauss-Newton estimation sequence for $\boldsymbol{\theta}$ becomes

$$\boldsymbol{\theta}^{(0)} = \begin{bmatrix} \cos^{-1}\left(\left(\mathbf{n}_{\mathrm{uc}}^{*}\right)_3 / \left|\mathbf{n}_{\mathrm{uc}}^{*}\right|\right) \\ \arctan_2\left(\left(\mathbf{n}_{\mathrm{uc}}^{*}\right)_2 / \left(\mathbf{n}_{\mathrm{uc}}^{*}\right)_1\right) \end{bmatrix} \tag{52}$$

$$\boldsymbol{\theta}^{(\mathrm{i})} = \boldsymbol{\theta}^{(\mathrm{i\text{-}1})} - \left(M^{\mathrm{T}}(\boldsymbol{\theta}^{(\mathrm{i\text{-}1})})\ \mathsf{F}\, M(\boldsymbol{\theta}^{(\mathrm{i\text{-}1})})\right)^{-1} M^{\mathrm{T}}(\boldsymbol{\theta}^{(\mathrm{i\text{-}1})})\left(\mathbf{G} + \mathsf{F}\hat{\mathbf{n}}(\boldsymbol{\theta}^{(\mathrm{i\text{-}1})})\right) \tag{53}$$

$$\boldsymbol{\theta}^{*} = \lim_{i\to\infty} \boldsymbol{\theta}^{(\mathrm{i})} \tag{54}$$

The matrix $M(\boldsymbol{\theta})$ as defined by equation (47) is singular if the spacecraft spin axis is collinear with the z-axis. Of course, the problem is easily solved by choosing a different set of polar and azimuthal axes for which new equations can be obtained. We accomplish this, effectively, by cyclic permutations of the indices in equations (43, 47, 52).

NUMERICAL RESULTS

All four algorithms described in this study are evaluated using two numerical examples that differ mainly in the levels of spin-axis observability.[§]

The first example, illustrating the case of good observability, uses an Earth-locked spacecraft in a 100-minute circular equatorial orbit with the spacecraft z-axis parallel to the spin-axis of the Earth (the inertial z-axis) and the spacecraft x-axis pointing toward the nadir. The spacecraft obtains data once per minute from a (coarse) sensor suite of three equally accurate sensors, a magnetometer, a vector Sun

[§] Both examples are selected for the purpose of generating a set of data and illustrating analytical findings of this study. Our conclusions remain valid regardless of the selected spin-rate which for many missions may be considerably higher. Our scenarios, however, are typical of missions from the 1970s and 1980s.

sensor, and an Earth-horizon sensor, each with angle-equivalent accuracy σ equal to 0.5 deg. For simplicity of simulations, we assume that the geomagnetic field at the equator is constant and is directed along the inertial z-axis

$$\hat{\mathbf{B}}_I = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{55}$$

and that the Sun direction remains inertially fixed and defined as

$$\hat{\mathbf{S}}_I = \begin{bmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{bmatrix} \tag{56}$$

where $\alpha = 23$ deg and where the subscript I denotes the inertial frame. We also assume that the Sun will be observable for one half of the orbit, specifically for longitude ℓ in the interval $-90 \text{ deg} \le \ell \le +90 \text{ deg}$. Thus individual sensor measurements

$$z_{B,k} = \hat{\mathbf{B}}_k^{\mathrm{T}}\, \hat{\mathbf{n}} + v_{B,k} \tag{57a}$$

$$z_{S,k} = \hat{\mathbf{S}}_k^{\mathrm{T}}\, \hat{\mathbf{n}} + v_{S,k} \tag{57b}$$

$$z_{E,k} = \hat{\mathbf{E}}_k^{\mathrm{T}}\, \hat{\mathbf{n}} + v_{E,k} \tag{57c}$$

are grouped accordingly

$$\mathbf{Z}_k = \begin{cases} \begin{bmatrix} z_{B,k} & z_{S,k} & z_{E,k} \end{bmatrix}^{\mathrm{T}} \text{ with } R_k = \sigma^2 I_{3\times 3} \text{ for } \ell \in [-90 \text{ deg}, +90 \text{ deg}] \\ \begin{bmatrix} z_{B,k} & z_{E,k} \end{bmatrix}^{\mathrm{T}} \text{ with } R_k = \sigma^2 I_{2\times 2} \text{ for } \ell \in [+90 \text{ deg}, +270 \text{ deg}] \end{cases} \tag{58ab}$$

where $\hat{\mathbf{E}}$ denotes the Earth nadir direction, and the subscript k denotes measurement frame number. Such a model is exceedingly simplified but will be adequate for highlighting the effects of the unit-norm constraint.

The results computed based on one full orbit of data are displayed in Table 1 and equations (59a-c). The table includes results for the Lagrange-multiplier method (Table 1A), the unconstrained brute-force method (Table 1B), the incremental-vector method (Table 1C), and the incremental-angle method (Table 1D).

$$\mathsf{F} = \begin{bmatrix} 1.231 & 0 & 0.241 \\ 0 & 0.650 & 0 \\ 0.241 & 0 & 1.415 \end{bmatrix} \times 10^6, \qquad \mathsf{G} = \begin{bmatrix} -0.241 \\ -0.001 \\ -1.416 \end{bmatrix} \times 10^6 \tag{59ab}$$

$$\mathsf{P} \equiv \mathsf{F}^{-1} = \begin{bmatrix} 0.841 & 0 & -0.143 \\ 0 & 1.538 & 0 \\ -0.143 & 0 & 0.731 \end{bmatrix} \times 10^{-6} \tag{59c}$$

Table 1A contains the three unnormalized components of the spin-vector and the Lagrange multiplier. The single iteration in Table 1B shows the result of the brute-force normalization of the spin-vector. Table 1C and Table 1D contain normalized components of the spin-vector as well as the magnitude of the difference between successive iterations. Note that the two incremental methods presented in these tables are initialized with the brute-force estimate. The iteration was terminated in each case when the last change in the result was smaller than 0.000001. It is important to emphasize that the fast convergence of all constrained estimation methods was the result of the good initial estimate $\hat{\mathbf{n}}_{\mathrm{uc}}$ provided by the unconstrained approximation. Without the use of $\hat{\mathbf{n}}_{\mathrm{uc}}$, as many as a dozen iterations were found to be necessary. The errors in n_3 are much smaller than the others due to the disproportionate influence of the norm constraint on this component given our choice of $\hat{\mathbf{n}}^{\mathrm{true}}$. Had we chosen $\hat{\mathbf{n}}^{\mathrm{true}}$ to be different from a coordinate axis, this would not have been the case.

Accuracies of the spin-axis attitude estimation for all four methods can be compared using the 1-σ confidence intervals calculated using their corresponding covariance matrices. Not surprisingly, all three properly constrained methods produce the same confidence intervals, and, given the relatively low correlations in P (Eq.(59c)), the unconstrained brute-force method performs only slightly worse. The large value of λ was also not unexpected based on the fact that according to equation (17b)

$$\lambda = 0 \pm 1170 \tag{60}$$

We designed the second example to evaluate performance of the algorithms for the case of a poorer observability. To this end, we reduced the orbit angular interval to one quarter of the orbit, i.e. $0\ \mathrm{deg} \le \ell \le 45\ \mathrm{deg}$, and reduced the measurement set so that

$$\mathbf{Z}_k = \begin{bmatrix} z_{E,k} & z_{S,k} \end{bmatrix}^{\mathrm{T}} \text{ with } R_k = \sigma^2 I_{2\times 2} \tag{61ab}$$

Other aspects of this example remained the same as in the previous example, including the number of measurements, 100, which was accomplished by adjusting the time interval between measurements appropriately.

The results, also organized in the same manner as in the previous example, are shown in Table 2 and equations (62a-c). Note that the three correctly constrained methods (Tables 2A, 2C, 2D) all yield the same result to six decimal places for the estimate of the spin-axis vector.

TABLE 1 COMPARISON FOR CASE OF GOOD OBSERVABILITY

A. Lagrange-Multiplier Method

iteration	n_1	n_2	n_3	λ
0	0.000241	0.000156	1.000103	0
1	0.000261	0.000156	0.999999	143
2	0.000261	0.000156	0.999999	143
	$\pm$ 0.000901	$\pm$ 0.001240	$\pm$ 0	

B. Unconstrained Brute-Force Method

iteration	n_1	n_2	n_3
0	0.000241	0.000156	0.999999
	$\pm$ 0.000917	$\pm$ 0.001240	$\pm$ 0

C. Incremental-Vector Method

iteration	n_1	n_2	n_3	$\lvert \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1} \rvert$
0	0.000241	0.000156	0.999999	–
1	0.000261	0.000156	0.999999	2.0×10^{-5}
2	0.000261	0.000156	0.999999	2.3×10^{-9}
	$\pm$ 0.000901	$\pm$ 0.001240	$\pm$ 0	

D. Incremental-Angle Method

iteration	n_1	n_2	n_3	$\lvert \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1} \rvert$
0	0.000241	0.000156	0.999999	–
1	0.000261	0.000156	0.999999	2.0×10^{-5}
2	0.000261	0.000156	0.999999	1.4×10^{-7}
	$\pm$ 0.000901	$\pm$ 0.001240	$\pm$ 0	

Performance of the unconstrained brute-force method is decidedly poorer. It results in the standard deviations of the estimates of n_1 and n_2 which are larger than those of the properly-constrained methods by 100% and 50%, respectively. The trace of the Cartesian spin-axis attitude covariance matrix for the brute-force estimate is larger than that for the properly-constrained estimate by a factor 2.3.

$$\mathsf{F} = \begin{bmatrix} 2.186 & 0.417 & 0.472 \\ 0.417 & 0.239 & 0 \\ 0.472 & 0 & 0.200 \end{bmatrix} \times 10^6, \qquad \mathbf{G} = \begin{bmatrix} -0.471 \\ 0.001 \\ -0.201 \end{bmatrix} \times 10^6 \tag{62ab}$$

$$\mathsf{P} \equiv \mathsf{F}^{-1} = \begin{bmatrix} 2.879 & -5.015 & -6.784 \\ -5.015 & 12.909 & 11.814 \\ -6.784 & 11.814 & 20.969 \end{bmatrix} \times 10^{-6} \tag{62c}$$

In this example, correlations in the unconstrained covariance matrix P (Eq.(62c)) $\rho_{12} = -0.862$, $\rho_{13} = -0.873$ and $\rho_{23} = 0.718$ are considerably higher than those in the previous example. The convergence of the iterative algorithms in this case is poorer but better than more than a dozen iterations, which would be the case without a good initial estimate.

When evaluating the computational efficiency of the proposed algorithms, we tested two implementations of the Lagrange-multiplier method. The first one followed equations (12) exactly including the matrix inverse of equation (12b). The second implementation employed an alternate iteration scheme in which linear-equation solvers were used to compute the two auxiliary vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined to be the solutions of

$$\left(\mathsf{F} + \lambda_{i-1} I_{3\times3}\right)\mathbf{V}_{1,i-1} = \mathbf{G} \qquad \text{and} \qquad \left(\mathsf{F} + \lambda_{i-1} I_{3\times3}\right)\mathbf{V}_{2,i-1} = \mathbf{V}_{1,i-1} \tag{63ab}$$

respectively. Then,

$$\mathbf{n}_{i-1}^{\mathrm{T}}\mathbf{n}_{i-1} = \mathbf{G}^{\mathrm{T}}\mathbf{V}_{2,i-1} = \mathbf{V}_{1,i-1}^{\mathrm{T}}\mathbf{V}_{1,i-1} \tag{64a}$$

$$\mathbf{n}_{i-1}^{\mathrm{T}} D_{i-1}\mathbf{n}_{i-1} = \mathbf{V}_{1,i-1}^{\mathrm{T}}\mathbf{V}_{2,i-1} \tag{64b}$$

and

$$\hat{\mathbf{n}}^* = \lim_{i\to\infty} \mathbf{V}_{1,i} \tag{64c}$$

TABLE 2 COMPARISON FOR CASE OF POOR OBSERVABILITY

A. Lagrange-Multiplier Method

iteration	n_1	n_2	n_3	λ
0	-0.000602	-0.002632	1.003112	0
1	0.000400	-0.004375	1.000010	148
2	0.000407	-0.004387	0.999990	149
3	0.000407	-0.004387	0.999990	149
	± 0.000828	± 0.002501	± 0	

B. Unconstrained Brute-Force Method

iteration	n_1	n_2	n_3
0	-0.000600	-0.002620	0.999996
	± 0.001697	± 0.003593	± 0

C. Incremental-Vector Method

iteration	n_1	n_2	n_3	$\lvert \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1} \rvert$
0	-0.000601	-0.002624	0.999996	–
1	0.000407	-0.004388	0.999990	0.0020
2	0.000407	-0.004388	0.999990	7.0×10^{-7}
	± 0.000828	± 0.002501	± 0	

D. Incremental-Angle Method

iteration	n_1	n_2	n_3	$\lvert \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1} \rvert$
0	-0.000601	-0.002624	0.999996	–
1	0.001180	-0.004017	0.999991	0.0022
2	0.000361	-0.004309	0.999991	0.00087
3	0.000408	-0.004387	0.999990	0.000091
4	0.000407	-0.004387	0.999990	7.0×10^{-7}
	± 0.000901	± 0.001240	± 0	

TABLE 3 EXECUTION TIMES FOR SPIN-AXIS ESTIMATION

Method	Relative Execution Time
Iterative Optimal Methods	
Lagrange-multiplier (matrix inverse)	76
Lagrange-multiplier (linear equation)	88
Incremental-vector	66
Incremental-angle	100
Non-iterative Approximate Method	
Brute-force	34

We devised the alternate implementation looking to exploit computational advantages that linear-equation solvers often have over matrix inversion algorithms. However, as the relative execution times summarized in Table 3 show, in Matlab® the original implementation of the Lagrange-multiplier method performed better. All five results are for the first numerical example and are based on execution of optimized Matlab® code. Each iterative method was terminated after a single iteration beyond the initial brute-force step (in the implementations of the Lagrange-multiplier method, equivalently, the estimate uses λ_1). Overall, the incremental-vector method would seem to offer the best value.

We checked consistency of our calculations by comparing the results for the model covariances, as given by equations (18), (28), (41), and (50), with the sampled covariance matrices for the spin-axis attitude estimate

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{\text{sampled}} \equiv \frac{1}{N}\sum_{m=1}^{N}\left(\hat{\mathbf{n}}_m^* - \hat{\mathbf{n}}^{\text{true}}\right)\left(\hat{\mathbf{n}}_m^* - \hat{\mathbf{n}}^{\text{true}}\right)^{\text{T}} \tag{65}$$

and with a similar definition for the sampled covariance matrix $P_{\hat{\mathbf{n}}_{\text{uc}}\hat{\mathbf{n}}_{\text{uc}}}^{\text{sampled}}$. Here, $\hat{\mathbf{n}}_m^*$ is the estimate of the spin-axis vector for the *m*-th sampled data set, $m = 1, \ldots, N$. The sampled covariance matrix is a random matrix that satisfies

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{\text{sampled}} = P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} + \Delta P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{\text{sampled}} \tag{66}$$

where for N very large $\left(\Delta P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{\text{sampled}}\right)_{ij}$ will be approximately Gaussian and zero-mean with variance given by

$$\text{Var}\left\{\left(P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{\text{sampled}}\right)_{ij}\right\} = \frac{1}{N}\left[\left(P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\right)_{ii}\left(P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\right)_{jj} + \left(P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\right)_{ij}^2\right] \tag{67}$$

We found agreement within the anticipated confidence bounds for all four spin-axis attitude estimation methods and both numerical examples for 100 sample tests ($N = 100$). As an example, for the iterative algorithms of the second numerical example we present

$$\begin{bmatrix} 0.707 & -1.530 \\ -1.530 & 7.549 \end{bmatrix} \times 10^{-6} = \begin{bmatrix} 0.685 & -1.193 \\ -1.193 & 6.253 \end{bmatrix} \times 10^{-6} \pm \begin{bmatrix} 0.116 & 0.231 \\ 0.231 & 0.500 \end{bmatrix} \times 10^{-6} \quad (68)$$

where the three matrices are in the same order as in equation (66), but where, for reasons of space, we have deleted the uninteresting third row and third column. The errors in the sampled covariances are 0.3σ, -1.7σ and 0.2σ, where σ is the appropriate standard deviation for each covariance.

CONCLUSIONS

Four algorithms have been presented for spin-axis attitude estimation, all of which estimate a spin-axis vector, which is a representation of the spacecraft spin-axis with respect to some reference (typically inertial) coordinate system. All but one of these algorithms produce solutions that account for the norm constraint of the spin-axis attitude, but the manner in which they account for it differs from algorithm to algorithm. The first method presented in this work estimates simultaneously all three components of the spin-axis vector directly and maintains the norm constraint explicitly by means of a Lagrange multiplier. The second method also estimates all three components simultaneously but imposes the constraint by brute force at the end of the calculation and is only approximate. The third and fourth methods estimate the spin-axis vector incrementally using different two-dimensional parameterizations, vectors and angles, to maintain the constraint to second order in the increment, which is driven to zero. Note that all of these algorithms are iterative with the exception of the brute-force method which simply generates a first approximation of the spin-axis vector estimate and is, in fact, used to initialize the iterative methods.

The main conclusion of this study is that the norm constraint must be taken into account properly in the estimation algorithm and not simply applied to the unconstrained estimate at the end of the calculation. In our numerical examples, we found the unconstrained estimate to be an excellent first approximation. In fact, in the first example that lacked significant correlations, the unconstrained estimate when normalized provided all the accuracy that was needed. However, in the presence of significant correlations in the second example, the brute-force method produced decidedly poorer results. We can support this finding by a simple analytical exercise in which we write the inverse unconstrained Fisher information matrix and the true spin-axis vector as

$$\mathsf{P} \equiv \mathsf{F}^{-1} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{n}}^{\text{true}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (69ab)$$

Then it follows from equations (18), (19), (28), and (29) that

$$P_{\hat{\mathbf{n}}\hat{\mathbf{n}}} = \begin{bmatrix} a - e^2/c & d - ef/c & 0 \\ d - ef/c & b - f^2/c & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad P_{\hat{\mathbf{n}}_{\text{uc}}\hat{\mathbf{n}}_{\text{uc}}} = \begin{bmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (70ab)$$

from which it is evident that the variances of the components of the correctly constrained spin-axis vector are smaller in general and the difference is more pronounced for larger correlations.

Another way to see the importance of constraint is by computing the figure of merit

$$\mu(\hat{\mathbf{m}}) \equiv \left(\hat{\mathbf{n}}^* - \hat{\mathbf{m}}\right)^{\mathrm{T}} P_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{-1} \left(\hat{\mathbf{n}}^* - \hat{\mathbf{m}}\right) \tag{71}$$

When this quantity is computed for $\hat{\mathbf{m}} = \hat{\mathbf{n}}^{\text{true}}$, we expect the result to have a χ^2 distribution for two degrees of freedom (since $\hat{\mathbf{n}}$ has only two degrees of freedom). In a test for the first example with 100 sample orbits we found a sampled mean of 2.07 and a sampled standard deviation of 1.87, both of which are consistent with mean 2 and variance 4 expected from the distribution. However, for a good initial guess of the spin-axis vector, we expect to have much smaller values. Thus, we can test the accuracy of the brute-force estimate by computing $\mu(\hat{\mathbf{m}})$ at $\hat{\mathbf{m}} = \hat{\mathbf{n}}_{\text{uc}}$. For the first example, we have found a sampled mean of 0.0015, which indicates that the brute-force estimate serves as a good initial guess and differs from the true value by terms only of order σ^2. On the other hand, in a similar test for the second example, we have found the sampled value of $\mu(\hat{\mathbf{n}}^{\text{true}})$ to be 1.73, compared to an anticipated value of 2 ± 0.2, and $\mu(\hat{\mathbf{n}}_{\text{uc}})$ to be 4.15, the very large value (eleven standard deviations from the expectation) demonstrating an unacceptable level of accuracy given the accuracy of the data.

Note that the error bounds in Tables 1 and 2 are meaningful despite the fact that both the 3×3 Cartesian spin-axis covariance and Fisher information matrices with the constraint are of rank 2. The error bounds reflect the true variation of the estimates, even though they provide no information on the correlation of individual components of the estimate.

We see that there are indeed interesting similarities between spin-axis and three-axis attitude estimation [2]. For example, constrained optimization carried out in the Lagrange-multiplier method is analogous to that of the QUEST algorithm [5] and to similar techniques found in later fast solutions to the Wahba problem [6] like FOAM and ESOQ [7]. Both batch least-square [2] and Kalman filter [8] methods of three-axis attitude estimation use the idea of incremental parameterization that is also employed by the two tangent plane methods of spin-axis estimation.

Similarities can be found not only in the mathematical techniques, but also in the structures of spin-axis and three-axis attitude estimators. For example, the 3×3 Cartesian spin-axis attitude covariance can be regarded as representing both the spin axis attitude (at least, within a sign) and its covariance matrix in the same way that the attitude profile matrix B does for the three-axis attitude estimation [4]. We find additional similarities when we compare the spin-axis attitude and the direction-cosine matrix (DCM) estimators. Both employ linear measurement models which, in the presence of the Gaussian measurement noise, result in similar quadratic cost functions [2, 9] that assure global convergence and computational efficiency of the estimators (no need to re-compute quantities like F and G in analogy to the attitude profile matrix for the Wahba problem).

We have found that, of the four presented algorithms, the incremental vector method demonstrates the best performance. It converges in a smaller number of iterations than the other iterative methods and ensures better accuracy than the brute-force method. The Lagrange-multiplier method, however, although slightly slower, has the advantage of being able to treat cases in which F is only of rank 2. We have also found that, implemented in Matlab®, a single iteration of the incremental method took the shortest amount of time compared to the other methods (Table 3). Although, execution times may not be very significant in an interpreted language such as Matlab®, this result appears reasonable due to a likely greater computational burden of trigonometric functions and 3×3 matrix inversion found in the incremental angle and the Lagrange-multiplier methods, respectively.

The results of the present work will be presented in more detail in a later publication [10].

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