MINIMUM ORDER ADAPTIVE ATTITUDE CONTROL

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The paper introduces a method for generating identical attitude error trajectories for a rigid-body with respect to different inertial and non-inertial frames. The method is formalized via the application of the trajectory dependent operator that acts on the input torque and maps it into a new torque such that the attitude error trajectory is preserved but with respect to a new reference frame. Consequently, this result establishes duality of attitude stabilization and tracking control and provides a unified framework for their analysis and design. Within this framework, control laws dependency on the inertia matrix is investigated, which leads to the introduction of a minimum order adaptive re-design methodology. All the development in this paper is done independently from any specific attitude representation.

INTRODUCTION

Attitude dynamics and control of a rigid body have been studied by many authors from both theoretical and practical standpoint.[1-8] Rigid-body dynamics represent one of the classical examples of passive nonlinear systems linear in control, which can be stabilized by a very simple linear or almost linear feedback control laws.[3-5] Rigid-body kinematics is also a passive nonlinear system driven by the dynamics. For this reason, the attitude itself can also be stabilized by a simple linear feedback control laws. [3-5] What is more, passive nature of both dynamics and kinematics permits angular velocity measurements to be replaced by the outputs of the stable linear system driven by the time rate of change of the chosen attitude representation.[3,5] While attitude tracking appears intuitively to be a natural extension of the stabilization problem, the nonlinear nature of attitude kinematics and dynamics obscured the exact mechanism of such extension. This paper seeks to clarify this issue and, in doing so, to point out duality between attitude motions with respect to different reference frames. The approach proposed in this paper is parallel to that of the structured dynamic model inversion[8] in the sense that both seek to reconstruct the input that produces desired dynamics. The goal of the approach presented in this paper is *not* to produce any selected dynamics, but rather to preserve the attitude error dynamics with respect to arbitrary moving frames. This can be accomplished by a *linear* operation on the input torque. This original input torque can itself be a result of the control law execution designed to ensure desired error dynamics with respect to a certain reference trajectory. Application of the proposed linear operation will ensure that this control law can be extended to tracking other types of trajectories subject to obvious practical limitations (e.g. sufficient authority, bandwidth, etc.). This becomes possible because both the operation itself is linear and the attitude dynamics are input-linear. The definition of the operation is nonlinear in reference angular velocity and acceleration and linear in the inertia matrix. The latter fact is exploited in the development of straightforward model reference adaptive control (MRAC) extension of the original operation. This way the original inertia matrix independent stabilizing control law can be used for tracking without certain knowledge of inertia matrix. Different forms of adaptation are presented that require different number of additional variables to be integrated depending on the type of the original control law.

Nomenclature:

A cross-product of two three-dimensional vectors $\mathbf{a}\times\mathbf{b}$ can be represented as the matrix-vector product $\mathbf{a}^{\times}\mathbf{b}$, where $\mathbf{a}^{\times} \stackrel{\Delta}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \Re^{3\times 3}$ is the skew-symmetric matrix constructed from the elements of vector $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$, where $\mathbf{a},\mathbf{b} \in \Re^3$ and both are expressed in the frame $\{\hat{\mathbf{e}}_\mathbf{i}\}$, which denotes a triad of mutually orthogonal unit vectors.

There are three types of frames used in this paper. One is the *body frame* $\{\hat{\mathbf{b}}_\mathbf{i}\}$ attached to a rotating rigid body. Another is the *reference* or *desired frame* $\{\hat{\mathbf{r}}_\mathbf{i}\}$, which is always a frame with respect to which attitude errors are found. Different reference frames used in this paper are denoted with subscripts r or R (e.g. $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$) in general and with subscript O (e.g. $\{\hat{\mathbf{r}}_\mathbf{i}\}_O$) if the

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frame is inertial. All the variables referred to this frame are similarly denoted with the subscript. Finally, the *inertial frame* $\left\{\hat{\mathbf{i}}_\mathbf{i}\right\}$ provides a common inertial reference for all of other frames. Unless stated otherwise, it is assumed that all variables referring to the motion of the reference frame with respect to the inertial frame are denoted with a "bar" (e.g. $\overline{\mathrm{w}}$), all variables referring to the body motion with respect to the inertial frame use capital letters (e.g. W) and all variables referring to the motion of the body frame with respect to the reference frame are also called attitude *error variables* and do not have a special designation (e.g. w). The variables commonly found in any of these categories are some attitude representation $\mathbf{s}$ belonging to a group isomorphic to $SO(3)$[9,10] and angular velocity $\mathrm{w} \in \Re^3$, which is always expressed in the frame, motion of which it describes. Note that differentiation of the angular velocity with respect to time $\dot{\mathrm{w}} \in \Re^3$ is also carried out in this frame. Additional notation includes transformation to a rotation matrix $\mathbf{C(s)} \in SO(3)$, the exact form of which depends on the attitude representation $\mathbf{s}$. Also, depending on this representation are the identity attitude $\mathbf{1}_\mathbf{s}$ and the inverse attitude $\mathbf{s}^{-1}$, such that $\mathbf{s}^{-1} \circ \mathbf{s} = \mathbf{s} \circ \mathbf{s}^{-1} = \mathbf{1}_\mathbf{s}$, where $\circ$ denotes this group's composition operation. The symmetric positive definite and constant inertia matrix $\mathbf{I} \in \Re^{3\times3} : \mathbf{I} = \mathbf{I}^\mathbf{T} > 0$ is defined in the body frame. The identity operator is denoted by E. Attitude trajectories and torque are assumed to be functions of time $t \geq t_0$ unless stated otherwise. Their initial values specified at time t_0 are denoted with subscript "0". The attitude trajectory of the body with respect to a reference frame $\left\{\hat{\mathbf{r}}_\mathbf{i}\right\}_r$ is denoted by a pair $(\mathbf{s}, \mathrm{w})$. The attitude trajectory $(\mathbf{S}, \mathrm{W})$ with respect to the inertial frame $\left\{\hat{\mathbf{i}}_\mathbf{i}\right\}$ is equivalently denoted by another pair $(\mathbf{s}, \mathrm{w})_\mathbf{r}$, which implies that

$$\mathbf{S} = \mathbf{s} \circ \overline{\mathbf{s}}_\mathbf{r}, \qquad (1)$$

$$\mathrm{W} = \mathrm{w} + \mathbf{C(s)}\overline{\mathrm{w}}_\mathbf{r}. \qquad (2)$$

Rigid body rotational kinematics and dynamics are described by

$$\dot{\mathbf{S}} = \mathbf{h}(\mathbf{S}, \mathrm{W}), \qquad (3)$$

$$\mathbf{I}\dot{\mathrm{W}} = -\mathrm{W}^\times \mathbf{I}\mathrm{W} + \mathbf{g}(\mathbf{S}, \mathrm{W}), \qquad (4)$$

where the exact form of kinematics $\mathbf{h}(\mathbf{s}, \mathrm{w})$ depends on the particular attitude representation, $\mathbf{s}$, and $\mathbf{g}(\mathbf{S}, \mathrm{W}) \in \Re^3$ is the net applied torque expressed in the body frame. All contributions to the net applied torque are always expressed in the body frame. The rotational kinematics and dynamics can equivalently be re-written in terms of $(\mathbf{s}, \mathrm{w})$ and $(\overline{\mathbf{s}}_\mathbf{r}, \overline{\mathrm{w}}_\mathbf{r})$ with respect to some reference frame $\left\{\hat{\mathbf{r}}_\mathbf{i}\right\}_r$:[7]

$$\dot{\mathbf{s}} = \mathbf{h}(\mathbf{s}, \mathrm{w}), \qquad (5)$$

$$\mathbf{I}\dot{\mathrm{w}} = -\mathrm{W}^\times \mathbf{I}\mathrm{W} - \mathbf{I}\left[\dot{\overline{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r} + \mathbf{g}(\mathbf{s}, \mathrm{w})_\mathbf{r}, \qquad (6)$$

where $\left[\dot{\overline{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r} \overset{\Delta}{=} \mathbf{C(s)}\dot{\overline{\mathrm{w}}}_\mathbf{r} - \mathrm{W}^\times \mathbf{C(s)}\overline{\mathrm{w}}_\mathbf{r}$. These equations are said to describe *error kinematics and dynamics* (or simply error dynamics, for brevity) and trajectories $(\mathbf{s}, \mathrm{w})$ generated by them are called *error trajectories*.

Additional nomenclature is introduced as needed throughout this paper.

DUALITY OF ATTITUDE MOTIONS IN INERTIAL AND MOVING FRAMES

This section establishes that the attitude motion of a rigid body with respect to an inertial frame can be mapped one-to-one onto the attitude motion with respect to an arbitrary moving frame. This mapping is mechanized via the appropriate re-formulation of the net applied torque. The operator that defines this mapping is presented in the following theorem.

Theorem 1:

Let the attitude error trajectory $(\mathbf{s}, \mathrm{w})$ defined with respect to the inertial reference frame $\left\{\hat{\mathbf{r}}_\mathbf{i}\right\}_\mathbf{O}$ be driven by the net applied torque $\mathbf{g}(\mathbf{s}, \mathrm{w})_\mathbf{O}$. Then reference trajectory dependent linear shift operator

$$\overline{L}\left\{\Re^3 \times \Re^3\right\} : \Re^3 \to \Re^3$$

$$\overline{L}_r \overset{\Delta}{=} \overline{L}\left\{\overline{\mathrm{w}}_\mathbf{r}, \dot{\overline{\mathrm{w}}}_\mathbf{r}\right\} \overset{\Delta}{=} E + \mathrm{W}^\times \mathbf{I}\mathrm{W} - \mathrm{w}^\times \mathbf{I}\mathrm{w} + \mathbf{I}\left[\dot{\overline{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r}, \ \forall \overline{\mathrm{w}}_{\mathbf{0r}}, \dot{\overline{\mathrm{w}}}_\mathbf{r} \in \Re^3 \qquad (7)$$

acting on the net applied torque $\mathbf{g}(\mathbf{s}, \mathrm{w})_\mathbf{O}$ results in a new applied torque $\mathbf{g}_\mathbf{r}(\mathbf{s}, \mathrm{w})_\mathbf{O} \overset{\Delta}{=} \overline{L}_r \mathbf{g}(\mathbf{s}, \mathrm{w})_\mathbf{O}$, such that the error trajectory $(\mathbf{s}, \mathrm{w})$ is preserved but with respect to the moving reference frame $\left\{\hat{\mathbf{r}}_\mathbf{i}\right\}_r$ provided the same initial attitude error conditions $(\mathbf{s}_\mathbf{0}, \mathrm{w}_\mathbf{0})$. The short-hand notation "$\overline{L}_r$" used above and later in this paper should be read as follows: "the operator $\overline{L}$ defined by the angular velocity trajectory of the reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$, $(\overline{\mathrm{w}}_\mathbf{r}, \dot{\overline{\mathrm{w}}}_\mathbf{r})$." Another short-hand notation "$\mathbf{g}_\mathbf{r}(\mathbf{s},\mathrm{w})_\mathbf{O}$" should be read as follows: "net applied torque generated via operator $\overline{L}_r$ from the torque $\mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{O}$ originally applied to the body so as to drive it along the attitude trajectory $(\mathbf{s},\mathrm{w})_\mathbf{O}$.

Proof:

Since the original reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_\mathbf{O}$ is inertial, $\overline{\mathrm{w}} = \dot{\overline{\mathrm{w}}} = \dot{\overline{\mathrm{w}}}_\mathbf{B} \equiv \mathbf{0}$, $\mathrm{w} \equiv \mathrm{W}$, Eqs.(5,6) can be re-written as

$$\dot{\mathbf{s}} = \mathbf{h}(\mathbf{s},\mathrm{w}), \tag{8}$$

$$\mathbf{I}\dot{\mathrm{w}} = -\mathrm{w}^\times \mathbf{I}\mathrm{w} + \mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{O}. \tag{9}$$

Then applying the new net torque $\mathbf{g}_\mathbf{N}(\mathbf{s},\mathrm{w})_\mathbf{O}$ drives the attitude error with respect to the moving reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$ according to

$$\mathbf{I}\dot{\mathrm{w}} = -\mathrm{W}^\times \mathbf{I}\mathrm{W} - \mathbf{I}\left[\dot{\overline{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r} + \mathbf{g}_\mathbf{r}(\mathbf{s},\mathrm{w})_\mathbf{O}, \tag{10}$$

which reduces to the form identical to Eq.(9) by inspection. Thus, it is shown that the attitude error kinematics and dynamics are governed by the same differential equations with respect to both reference frames, $\{\hat{\mathbf{r}}_\mathbf{i}\}_\mathbf{O}$ and $\{\hat{\mathbf{r}}_\mathbf{i}\}_\mathbf{r}$. Provided the same initial conditions $(\mathbf{s}_\mathbf{0}, \mathrm{w}_\mathbf{0})$, attitude error trajectories $(\mathbf{s},\mathrm{w})$ will also be identical with respect to both reference frames based on the existence and uniqueness theorem for solutions of ordinary differential equations.[11] This result was achieved via the application of the operator $\overline{L}_r$ as described in *Theorem 1*. Q.E.D.

Corollary 1:

The mapping defined by the operator $\overline{L}_r$ introduced in the previous theorem (Eq.(7)) is invertible and the inverse is defined by $\overline{L}_r^{-1}$:

$$\overline{L}_r^{-1} \stackrel{\Delta}{=} E - \mathrm{W}^\times \mathbf{I}\mathrm{W} + \mathrm{w}^\times \mathbf{I}\mathrm{w} - \mathbf{I}\left[\dot{\overline{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r}. \tag{11}$$

Proof:

The existence of the inverse mapping follows from the existence of the identity mapping $E \stackrel{\Delta}{=} \overline{L}\{\mathbf{0},\mathbf{0}\}$ obtained by inspection of Eq.(7). Also, by inspection, it can be shown that

$$\overline{L}_r^{-1}\overline{L}_r = \overline{L}_r\overline{L}_r^{-1} = E, \forall r. \tag{12}$$

Q.E.D.

Corollary 2:

The operator $\overline{L}_r$ reduces to identity when defined by any inertial reference frame. Consequently, the operator is invariable with respect to any inertial reference frame.

The results of *Theorem 1* can be extended to the case of two non-inertial reference frames with the help of *Corollary 1*.

Theorem 2:

Let the attitude error trajectory $(\mathbf{s},\mathrm{w})$ defined with respect to the non-inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ be driven by the net applied torque $\mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{R}$. Then reference trajectories dependent linear shift operator

$$L\left\{\Re^3 \times \Re^3 \times \Re^3 \times \Re^3\right\} : \Re^3 \to \Re^3$$

$$L_R^r \stackrel{\Delta}{=} \overline{L}_r \overline{L}_R^{-1} \tag{13}$$

acting on the net applied torque $\mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{R}$ will result in a new applied torque $\mathbf{g}_\mathbf{r}(\mathbf{s},\mathrm{w})_\mathbf{R} \stackrel{\Delta}{=} L_R^r \mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{R}$, such that the attitude error trajectory $(\mathbf{s},\mathrm{w})$ is preserved but with respect to another moving reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$ provided the same initial error conditions $(\mathbf{s}_\mathbf{0}, \mathrm{w}_\mathbf{0})$.

Proof:

The result follows immediately from the successive application of *Theorem 1* and *Corollary 1* to map from the non-inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ to the inertial frame and, using *Theorem 1* again, from the inertial frame to the non-inertial frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$.

Remark 1:

It follows from *Theorem 2* that, using the mapping Eq.(13), it is always possible to generate identical attitude error trajectories $(\mathbf{s},\mathrm{w})$, provided that the initial conditions $(\mathbf{s_0}, \omega_\mathbf{0})$ match. The latter requirement is not needed if the entire families of solutions of corresponding differential equations are compared. In this case, matching of two families of trajectories amounts to matching of the generating differential equations. It will be said that the attitude error dynamics are identical in this case.

Remark 2:

It should be emphasized that the function $\mathbf{g}(\mathbf{s}, \omega)_\mathbf{R}$ itself is *not* transformed by the operators. In other words, it is still evaluated along the original attitude trajectory as if the body attitude motions were described by the error trajectories $(\mathbf{s}, \omega)$ with respect to the frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ and not $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$.

Both remarks will be explored in the next section that studies the problems of attitude stabilization and tracking.

DUALITY OF ATTITUDE STABILIZATION AND TRACKING

The problem of stabilizing the attitude of a rigid body is an example of a problem known as a regulator problem in the general control literature. The control law solving this problem seeks to drive the error trajectory to the origin. Note that another type of control problem, known as a set point control, is easily converted to a regulator problem for the attitude motion.[3] Indeed, given any attitude representation $\mathbf{s}'$ and its desired constant value $\mathbf{s_d}$, the set point control problem is converted to a regulator problem via change of variables, for example:

$$\mathbf{s} \overset{\Delta}{=} \mathbf{s}' \circ \mathbf{s_d}^{-1}. \qquad (14)$$

Thus, both attitude stabilization and attitude set point control are treated within the framework of the regulator problem.

The attitude regulator problem is solved by a control law that drives the attitude error trajectory $(\mathbf{s}, \omega)$ to $(\mathbf{1_s}, \mathbf{0})$. If this objective is achieved for any initial conditions $(\mathbf{s_0}, \omega_\mathbf{0})$ then the control law is globally stabilizing.

The attitude tracking problem can also be viewed as the problem of driving $(\mathbf{s}, \omega)$ to $(\mathbf{1_s}, \mathbf{0})$, however, the attitude error trajectories are no longer defined with respect to an inertial frame, but with respect to some desired non-inertial frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$.

The following theorem describes relationship between these two types of control laws.

Theorem 3:

In the absence of disturbances, any control law stabilizing attitude in the error domain Π, i.e. $\forall(\mathbf{s_0}, \omega_\mathbf{0}) \in \Pi$ with $(\mathbf{s}, \omega)$ defined with respect to some inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_\mathbf{O}$, can be used to generate tracking control law via the application of the operator $\overline{L}_r$. The resulting control law preserves the error dynamics and domain Π, but with respect to the desired non-inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$.

Proof:

The proof follows directly form the *Theorem 1*, since Eq.(8) is shared by both definitions of attitude error and Eq.(9) and Eqs.(10) were shown to be identical with the use of the operator $\overline{L}_r$. Q.E.D.

Entirely similar approach based on *Theorem 2* leads to the following theorem given without proof.

Theorem 4:

In the absence of disturbances, any control law tracking a reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ in the attitude error domain Π, i.e. $\forall(\mathbf{s_0}, \omega_\mathbf{0}) \in \Pi$, can be used to generate tracking control law via the application of the operator L_R^r. The resulting control law will preserve the error dynamics and domain Π, but with respect to the desired non-inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$.

Corollary 3:

Theorems 3 and *4* can also be re-formulated for the cases of globally stabilizing and tracking control laws, respectively. The re-formulation is straightforward and is omitted for brevity.

Remark 3:

In this paper, a disturbance is defined as an unknown attitude dependent torque. The absence of disturbances is needed in order to make control generation via proposed operators realizable. The reason becomes clear from *Remark 2*, which states that the net applied torque must be evaluated along the original attitude trajectory. This is realizable only if all contributors of torque are either known or do not depend on the attitude. To this end, the net applied torque is presented in the following form:

$$\mathbf{g}(\mathbf{s},\omega)_{\mathbf{R}} = \mathbf{u}(\mathbf{s},\omega)_{\mathbf{R}} + \tau(\mathbf{s},\omega)_{\mathbf{R}} + \mathbf{p} + \mathbf{d}(\mathbf{s},\omega)_{\mathbf{R}} \quad (15)$$

where $\mathbf{u} \in \Re^3$ is the torque contribution from the active control, $\tau \in \Re^3$ is the known attitude dependent uncontrolled torque, $\mathbf{p} \in \Re^3$ is the attitude independent uncontrolled torque, $\mathbf{d} \in \Re^3$ is the disturbance or the unknown attitude dependent uncontrolled torque. The goal is to preserve all of these contributions while the actual reference trajectory is changed from $(\mathbf{s},\omega)_{\mathbf{R}}$ to $(\mathbf{s},\omega)_{\mathbf{r}}$. The active control $\mathbf{u}$ can be preserved by construction. The known attitude dependent uncontrolled torque τ can be compensated via the following augmented active control

$$\mathbf{u}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}} \stackrel{\Delta}{=} \mathbf{u}(\mathbf{s},\omega)_{\mathbf{R}} + \tau(\mathbf{s},\omega)_{\mathbf{R}} - \tau(\mathbf{s},\omega)_{\mathbf{r}} \quad (16)$$

Of course, the attitude independent uncontrolled torque $\mathbf{p}$ is preserved by definition. Thus, it is cleat that, if $\mathbf{d} \equiv \mathbf{0}$, the net applied torque is preserved along the new trajectory:

$$\mathbf{g}(\mathbf{s},\omega)_{\mathbf{R}} = \mathbf{u}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}} + \tau(\mathbf{s},\omega)_{\mathbf{r}} + \mathbf{p} \quad (17)$$

However, in the presence of disturbances, the difference between the net torque $\mathbf{g}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}}$ achievable along the new trajectory, i.e. the argument for the operator L^r_R, and the net torque driving the original trajectory, $\mathbf{g}(\mathbf{s},\omega)_{\mathbf{R}}$,

$$d\mathbf{g}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}} = \mathbf{d}(\mathbf{s},\omega)_{\mathbf{r}} - \mathbf{d}(\mathbf{s},\omega)_{\mathbf{R}} \quad (18)$$

cannot be eliminated.

In practice, the disturbances are always present, so the control performance is now re-evaluated using the concept of Bounded-Input-Bounded-Output (BIBO) stability.

Theorem 5:

Any attitude stabilizing control law that guarantees exponential stability[12] of the error dynamics in the domain Π, i.e. $\forall(\mathbf{s_0},\omega_{\mathbf{0}}) \in \Pi$ with $(\mathbf{s},\omega)$ defined with respect to some inertial reference frame $\{\hat{\mathbf{r}}_{\mathbf{i}}\}_{\mathbf{O}}$, can be used to generate tracking control law that guarantees BIBO stability in the same error domain Π for any desired non-inertial reference frame $\{\hat{\mathbf{r}}_{\mathbf{i}}\}_r$ via the application of the operator $\overline{L}_r$.

This theorem can be viewed as a special case of the next theorem that deals with the case of two tracking control laws. Hence the proof is deferred until *Theorem 6*.

Theorem 6:

Any attitude tracking control that guarantees exponential stability of the error dynamics in the absence of disturbances in the domain Π, i.e. $\forall(\mathbf{s_0},\omega_{\mathbf{0}}) \in \Pi$ with $(\mathbf{s},\omega)$ defined with respect to the non-inertial reference frame $\{\hat{\mathbf{r}}_{\mathbf{i}}\}_R$, can be used to generate tracking control law that guarantees BIBO stability in the same error domain Π for any desired non-inertial reference frame $\{\hat{\mathbf{r}}_{\mathbf{i}}\}_r$ via the application of the operator L^r_R.

Proof:

The preceding results indicate that the equations generated for the error dynamics with respect to $\{\hat{\mathbf{r}}_{\mathbf{i}}\}_r$ with the use of the operator L^r_R and the augmented active control $\mathbf{u}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}}$ will match the original equations but with the additional input $d\mathbf{g}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}}$. Hence, the problem can be viewed as driving the exponentially stable system with this input $d\mathbf{g}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}}$. This implies that the new system is BIBO stable.[12] Q.E.D.

Remark 4:

The additional input $d\mathbf{g}(\mathbf{s},\omega)^{\mathbf{r}}_{\mathbf{R}}$ is bounded if both disturbances, $\mathbf{d}(\mathbf{s},\omega)_{\mathbf{r}}$ and $\mathbf{d}(\mathbf{s},\omega)_{\mathbf{R}}$, are bounded in the attitude domains mapped from Π by their respective trajectories.

Remark 5:

Similar BIBO stability can be shown even without the use of the augmented control (Eq.(16)) for compensation of the known uncontrolled torque. Then, exponential stability in the absence of any uncontrolled torque demonstrated with respect to the reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ under the active control $\mathbf{u}(\mathbf{s},\omega)_\mathbf{R}$ implies BIBO stability with respect to any other reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$ with the use of the new active control $L_R^r \mathbf{u}(\mathbf{s},\omega)_\mathbf{R}$.

The results are also easily extended for global stability.

Corollary 4:

Theorems 5 and *6* can also be re-formulated for the cases of global exponentially and BIBO stable control laws. As in *Corollary 3*, the re-formulation is straightforward and is omitted for brevity.

The last result of this section concerns the dependency of an attitude control law on the inertia matrix.

Theorem 7:

Any attitude tracking control law that globally guarantees the identical error dynamics in the absence of disturbances with respect to any arbitrary non-inertial frame must necessarily depend at least linearly on the inertia matrix $\mathbf{I}$.

Proof:

Suppose there exists such control law $\mathbf{g}' \in \Re^3$ that does not depend on the inertia matrix $\mathbf{I}$. Let $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ and $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$ be the arbitrary non-inertial reference frames. According to the conditions of this theorem, the control law must globally guarantee the identical error dynamics in the absence of disturbances with respect to both of these frames. The results of *Theorem 4* and *Corollary 4* suggest the existence of control laws generated via $L_R^r \mathbf{g}'$ and $L_r^R \mathbf{g}'$, which also achieve this objective. Thus, it appears that there exist several torque, which, when applied to the same set of differential equations of the form Eqs.(5,6), produce identical error dynamics. Under the assumption that the error dynamics are controllable at least some time for some reference frames, it is inferred that different torque will produce different error trajectories $(\mathbf{s},\omega)$ starting with the same initial conditions $(\mathbf{s_0},\omega_\mathbf{0})$. Consequently, all torque must be identical in these cases, which, in turn, requires the operator L_R^r to become the identity operator E. Note that, as $\mathbf{g}'$ is said to be independent from the inertia matrix $\mathbf{I}$, the dependency on $\mathbf{I}$ must be eliminated using only the definition of the operator L_R^r itself. It can be seen by inspection of Eqs.(7,13) that this is not generally possible for arbitrary reference frames $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ and $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$. Hence, the contradiction is reached and $\mathbf{g}'$ must necessarily depend on the inertia matrix $\mathbf{I}$. The fact that the dependency is at least linear is easily established by considering such $\mathbf{g}'$ that it is independent from $\mathbf{I}$ for some reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_R$ and inspecting the form of $L_R^r \mathbf{g}'$. Q.E.D.

Remark 6:

The result of the last theorem provides an interesting insight into a so-called *reduction* property of certain control laws. Such control laws do not require knowledge of the inertia matrix for the special case of attitude stabilization.[7] The results presented in this paper indicate that each stabilizing control law, which is independent from the inertia matrix $\mathbf{I}$, establishes a corresponding tracking control law via the operator $\overline{L}_r$ that incorporates the inertia matrix linearly and possesses the reduction property.

The results of this section provide a basis for introduction of adaptive control as they show that unavoidable uncertainties in the inertia matrix and presence of disturbances will corrupt attitude tracking performance.

MINIMUM ORDER ADAPTIVE RE-DESIGN OF ATTITUDE TRACKING

The adaptive re-design addressed in this section deals with attitude tracking control laws that are generated via the adaptive operator $\tilde{\overline{L}}_r$ derived from $\overline{L}_r$ and acting on stabilizing control laws, which are independent from the inertia matrix.

Nomenclature:

The matrix $\mathbf{E_n} \in \Re^{n\times n}$ is the identity. The product of the inertia matrix $\mathbf{I} \overset{\Delta}{=} [I_{ij}], I_{ij} = I_{ji}, i, j = 1,2,3$ and the vector $\mathbf{a} \in \Re^3$ both expressed in $\{\hat{\mathbf{b}}_\mathbf{i}\}$ can be transformed into a different matrix-vector product $\mathbf{I}\mathbf{a} = \mathbf{a}^{\oplus}\overline{\mathbf{I}}$, where

$$\mathbf{a}^{\oplus} \overset{\Delta}{=} \begin{bmatrix} & a_2 & a_3 & 0 \\ diag\{\mathbf{a}\} & a_1 & 0 & a_3 \\ & 0 & a_1 & a_2 \end{bmatrix} \in \Re^{3\times 6}$$

and $\overline{\mathbf{I}} = [I_{11} \quad I_{22} \quad I_{33} \quad I_{12} \quad I_{13} \quad I_{23}]^\mathbf{T} \in \Re^6$. Any adaptive element is denoted with "~" and the difference between the adaptive and true values is denoted with "d". The adaptive re-design methodology is developed in this section under the assumption that all unknown torque contributions are constant, i.e. $\mathbf{d} \equiv \mathbf{0}$, $\mathbf{p} \equiv \mathbf{p_0}$. This also means that the operator $\tilde{\tilde{L}}_r$ acts only on the actively controlled contribution $\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O}$ of the net applied torque $\mathbf{g}(\mathbf{s},\mathrm{w})_\mathbf{O}$.

Several extensions of the proposed methodology, including adapting to bounded disturbances and operating on inertia dependent tracking control laws, are discussed in the next section.

The adaptive re-design depends on the type of the original Lyapunov function that can be found to demonstrate that the nominal control law $\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O}$ ensures asymptotic stability of the error dynamics in the absence of disturbances. Let Lyapunov function of Type 1 be radially unbounded, defined as

$$\begin{aligned} &V_{01}(\mathbf{s},\mathrm{w}) > 0, \dot{V}_{01}(\mathbf{s},\mathrm{w}) \le 0 \\ &\forall \mathbf{s} \ne \mathbf{1_s}, \forall \mathrm{w} \ne \mathbf{0} \end{aligned} \tag{19}$$

with $\dot{V}_{01}$ negative definite in $\mathbf{s}$ and such that its partial derivative with respect to w can be written as

$$\frac{\partial V_{01}(\mathbf{s},\mathrm{w})}{\partial \mathrm{w}} = \mathbf{w}_{\mathbf{01}}^{\mathbf{T}}(\mathbf{s},\mathrm{w})\mathbf{I}, \tag{20}$$

where $\mathbf{w_{01}}(\mathbf{s},\mathrm{w}) \in \Re^3$.

Let Lyapunov function of Type 2 also be radially unbounded, defined as

$$\begin{aligned} &V_{02}(\mathbf{s},\mathrm{w}) > 0, \dot{V}_{02}(\mathbf{s},\mathrm{w}) \le 0 \\ &\forall \mathbf{s} \ne \mathbf{1_s}, \forall \mathrm{w} \ne \mathbf{0} \end{aligned} \tag{21}$$

with $\dot{V}_{02}$ negative definite in $\mathbf{s}$. However, its partial derivative with respect to w can only be written as

$$\frac{\partial V_{02}(\mathbf{s},\mathrm{w})}{\partial \mathrm{w}} = \mathbf{w}_{\mathbf{02}}^{\mathbf{T}}(\mathbf{s},\mathrm{w}), \tag{22}$$

where $\mathbf{w_{02}}(\mathbf{s},\mathrm{w}) \in \Re^3$. In other words, the inertial matrix $\mathbf{I}$ cannot be extracted from the partial derivative as in the case of Type 1.

As was shown in the preceding section, the nominal operator $\bar{L}_r$ acting on the control law $\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O}$ generates a new control law, which, in the absence of uncontrolled torque, produces error dynamics identical to those produced by $\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O}$. Clearly, in this nominal case, stability can still be shown using the direct method applied to the same Lyapunov function. However, any realizable form of the operator $\bar{L}_r$ will have to use estimates of the true inertia matrix and as such will potentially be perturbed. Thus, the adaptive operator $\tilde{\tilde{L}}_r$ is sought such that, acting on the original control law, it ensures asymptotically vanishing attitude errors even in the presence of constant unknown torque $\mathbf{p_0}$. Modifications to the original Lyapunov functions are introduced to demonstrate this property. Note that the original operator $\bar{L}_r$ is linear and incorporates the inertia matrix linearly. In conjunction with the fact that rigid body dynamics are linear in torque, these results warrant the use of the standard MRAC methodology of replacing unknown constants with their time-varying counterparts. Depending on the type of the Lyapunov function different adaptive operators $\tilde{\tilde{L}}_{r1}$ and $\tilde{\tilde{L}}_{r2}$ are proposed.

If the original Lyapunov function is of Type 1, the adaptive operator $\tilde{\tilde{L}}_{r1}$ uses the time-varying inertia matrix $\tilde{\mathbf{I}}$ instead of the unknown true inertia $\mathbf{I}$ and the time-varying compensating torque $\tilde{\mathbf{q}} \stackrel{\Delta}{=} -\mathbf{p_0} + d\mathbf{q}$ instead of the unknown compensating torque $-\mathbf{p_0}$. With the use of the notation above, the adaptive operator $\tilde{\tilde{L}}_{r1}$ is defined as

$$\tilde{\tilde{L}}_{r1} \stackrel{\Delta}{=} E + \mathbf{F_r}\tilde{\mathrm{u}}, \tag{23}$$

where

$$\begin{aligned} &\mathbf{F_r} \in \Re^{3\times 9}, \mathbf{F_r} \stackrel{\Delta}{=} \\ &\left[\mathrm{W}^\times \mathrm{W}^\oplus - \mathrm{w}^\times \mathrm{w}^\oplus + \left[\dot{\bar{\mathrm{w}}}_\mathbf{B}\right]_\mathbf{r}^\oplus \quad \mathbf{E_3} \right], \end{aligned} \tag{24}$$

$$\tilde{\mathrm{u}} \stackrel{\Delta}{=} \begin{bmatrix} \bar{\tilde{\mathbf{I}}} \\ \tilde{\mathbf{q}} \end{bmatrix} \in \Re^9. \tag{25}$$

The perturbative torque is defined in the body frame as the difference between the results of the adaptive and nominal operators, $\tilde{\tilde{L}}_{r1}$ and $\bar{L}_r$, acting on the same original control law in the presence of the unknown torque $\mathbf{p_0}$:

$$\begin{aligned} d\mathbf{g_{r1}} &\stackrel{\Delta}{=} \tilde{\tilde{L}}_{r1}\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O} - \bar{L}_r\mathbf{u}(\mathbf{s},\mathrm{w})_\mathbf{O} + \mathbf{p_0} \\ &= \mathbf{F_r}d\mathrm{u} \end{aligned}. \tag{26}$$

The following theorem demonstrates adaptive re-design for systems with Type 1 Lyapunov function.

Theorem 8:

In the presence of the unknown inertia matrix $\mathbf{I}$ and constant torque $\mathbf{p_0}$, the adaptive operator $\tilde{\tilde{L}}_{r1}$ with the following adaptation law

$$\dot{\tilde{u}} = -G\mathbf{F_r^T}\mathbf{w_{01}}, \quad (27)$$

where $G \in \Re^{9\times 9}: G^{\mathbf{T}} = G > 0$, asymptotically stabilizes the error dynamics with respect to the reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}_r$ by operating on any inertia matrix independent and asymptotically stabilizing control law, stability of which can be shown with Type 1 Lyapunov function.

Proof:

Consider the following positive definite and radially unbounded candidate function

$$V_1(\mathbf{s}, w, du) = V_{01}(\mathbf{s}, w) + \frac{1}{2} du^{\mathbf{T}} G^{-1} du, \quad (28)$$

derived from the original Type 1 Lyapunov function. The derivative of the candidate function with respect to time evaluated along the trajectories of the closed-loop system is

$$\begin{aligned}\dot{V}_1(\mathbf{s}, w, du) &= \dot{V}_{01}(\mathbf{s}, w) \\ &+ \mathbf{w_{01}^T}\mathbf{I}\mathbf{I}^{-1} d\mathbf{g_{r1}} + d\dot{u}^{\mathbf{T}} G^{-1} du\end{aligned} \quad (29)$$

The last two terms vanish with the use of adaptation law of Eq.(27) and recognizing $\dot{\tilde{u}} \equiv d\dot{u}$. Hence, $\dot{V}_1 \le 0$ and, with V_1 being radially unbounded, $\mathbf{s}, w, du \in \ell_\infty$. Also, $d\dot{u} \in \ell_\infty$ based on Eq.(27) provided that V_{01} is smooth in w. Since the original function $\dot{V}_{01}$ is negative definite in $\mathbf{s}$, the largest invariant set $\{(\mathbf{s}, w, du): \dot{V}_1 = 0\}$ must necessarily include $\mathbf{s} = \mathbf{1}_s$ with $\dot{\mathbf{s}} = \mathbf{0}$. Then it follows that $w = \mathbf{0}$ for any attitude representation $\mathbf{s}$. (Otherwise, non-zero angular velocity must not induce motion of the body frame, which is not possible). Hence, the largest invariant set is $\{(\mathbf{s}, w, du): \mathbf{s} = \mathbf{1}_s, w = \mathbf{0}\}$, which also leads to $d\dot{u} = \mathbf{0}$ based on Eq.(27). Thus, according to LaSalle's invariance principle[13], du and, consequently, $\tilde{u}$ are guaranteed to be bounded with $\tilde{u}$ asymptotically approaching some constant (but not necessarily true) value. The attitude error is guaranteed to be bounded and to vanish asymptotically. Q.E.D.

The results of the preceding theorem indicate that any stabilizing control law that permits Type 1 Lyapunov function can be re-designed to adapt to the unknown inertia matrix and constant torque by extending the dimension of the state space by no more than 9 with only 6 contributed from the inertia matrix adaptation. This number increases significantly if only the function of Type 2 can be used to prove stability of the original control law.

The adaptive operator $\tilde{\tilde{L}}_{r2}$ is defined as

$$\tilde{\tilde{L}}_{r2} \overset{\Delta}{=} E + \tilde{\mathbf{Q}}m_{\mathbf{r}}, \quad (30)$$

where the true unknown matrix $\mathbf{Q} \in \Re^{3\times 9}$ is

$$\mathbf{Q} \overset{\Delta}{=} [\mathbf{Q_1} \quad \mathbf{Q_2} \quad \mathbf{I}], \quad (31)$$

$$\mathbf{Q_1} \overset{\Delta}{=} \begin{bmatrix} 0 & I_{23} & -I_{23} \\ -I_{13} & 0 & I_{13} \\ I_{12} & -I_{12} & 0 \end{bmatrix}, \quad (32)$$

$$\mathbf{Q_2} \overset{\Delta}{=} \begin{bmatrix} I_{13} & -I_{12} & \Delta I_{32} \\ -I_{23} & \Delta I_{13} & I_{12} \\ \Delta I_{21} & I_{23} & -I_{13} \end{bmatrix}, \quad (33)$$

$$\Delta I_{ij} \overset{\Delta}{=} I_{ii} - I_{jj}, i, j = 1,2,3, \quad (34)$$

$$m_{\mathbf{r}} \overset{\Delta}{=} \left[m_{\mathbf{r1}}^{\mathbf{T}} \quad m_{\mathbf{r2}}^{\mathbf{T}} \quad [\dot{w}_{\mathbf{B}}]_{\mathbf{r}}^{\mathbf{T}} \right]^{\mathbf{T}} \in \Re^{9\times 1}, \quad (35)$$

$$m_{\mathbf{r1}} \overset{\Delta}{=} [l_1^2 \quad l_2^2 \quad l_3^2]^{\mathbf{T}}, \quad (36)$$

$$m_{\mathbf{r2}} \overset{\Delta}{=} [l_1 l_2 \quad l_1 l_3 \quad l_2 l_3]^{\mathbf{T}}, \quad (37)$$

$$l_i \overset{\Delta}{=} \Omega_i - w_i, i = 1,2,3. \quad (38)$$

The perturbative torque in this case changes to

$$\begin{aligned} d\mathbf{g_{r2}} &\overset{\Delta}{=} \tilde{\tilde{L}}_{r2}\mathbf{u}(\mathbf{s}, w)_{\mathbf{O}} - \bar{L}_r\mathbf{u}(\mathbf{s}, w)_{\mathbf{O}} + \mathbf{p_0} \\ &= d\mathbf{Q}m_{\mathbf{r}} + d\mathbf{q} \end{aligned}. \quad (39)$$

The adaptive re-design for systems with Type 2 Lyapunov function is also different and is described in the following theorem.

Theorem 9:

In the presence of the unknown inertia matrix $\mathbf{I}$ and constant torque $\mathbf{p_0}$, the adaptive operator $\tilde{\tilde{L}}_{r2}$ with the following adaptation law

$$\dot{\tilde{\mathbf{Q}}} = -a^{-1}\mathbf{w}_{\mathbf{02}}\mathrm{m}_{\mathbf{r}}^{\mathbf{T}}, \qquad (40)$$

$$\dot{\tilde{\mathbf{q}}} = -b^{-1}\mathbf{w}_{\mathbf{02}}, \qquad (41)$$

where $a, b \in \Re^{+}$, asymptotically stabilizes the error dynamics with respect to the reference frame $\left\{\hat{\mathbf{r}}_{\mathbf{i}}\right\}_r$ by operating on any inertia matrix independent and asymptotically stabilizing control law, stability of which can only be shown by Type 2 Lyapunov function.

Proof:

Consider the following positive definite and radially unbounded candidate function

$$\begin{aligned} V_2(\mathbf{s}, \mathrm{w}, \mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{q}) &= V_{02}(\mathbf{s}, \mathrm{w}) \\ &+ \frac{a}{2} tr\left\{\mathrm{d}\mathbf{Q}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{Q}\right\} + \frac{b}{2}\mathrm{d}\mathbf{q}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{q} \end{aligned} \qquad (42)$$

derived from the original Type 2 Lyapunov function. The derivative of the candidate function with respect to time evaluated along the trajectories of the closed-loop system is

$$\begin{aligned} \dot{V}_2(\mathbf{s}, \mathrm{w}, \mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{q}) &= \dot{V}_{02}(\mathbf{s}, \mathrm{w}) \\ &+ \mathbf{w}_{\mathbf{02}}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{g}_{\mathbf{r2}} \\ &+ a\,tr\left\{\mathrm{d}\dot{\mathbf{Q}}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{Q}\right\} + b\,\mathrm{d}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{q} \end{aligned} \quad . \qquad (43)$$

After some re-arrangement of terms utilizing the trace invariance under cyclic permutations[14], the derivative of the candidate function becomes

$$\begin{aligned} \dot{V}_2(\mathbf{s}, \mathrm{w}, \mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{q}) &= \dot{V}_{02}(\mathbf{s}, \mathrm{w}) \\ &+ tr\left\{\mathbf{I}^{-1}\mathrm{d}\mathbf{Q}\mathrm{d}\dot{\mathbf{Q}}^{\mathbf{T}}a + \mathbf{I}^{-1}\mathrm{d}\mathbf{Q}\mathrm{m}_{\mathbf{r}}\mathbf{w}_{\mathbf{02}}^{\mathbf{T}}\right\} \\ &+ \mathbf{w}_{\mathbf{02}}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{q} + b\,\mathrm{d}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{I}^{-1}\mathrm{d}\mathbf{q} \end{aligned} \qquad (44)$$

and all the terms except the original derivative $\dot{V}_{02}$ vanish with the use of the adaptation law of Eqs.(40,41) and recognizing that $\dot{\tilde{\mathbf{Q}}} \equiv \mathrm{d}\dot{\mathbf{Q}}$, $\dot{\tilde{\mathbf{q}}} \equiv \mathrm{d}\dot{\mathbf{q}}$. The rest of the proof and its results parallel those of *Theorem 8*. Q.E.D.

Hence, the dimension of the state space for adaptation of stabilizing control laws that only permit Type 2 Lyapunov functions is increased by 21 compared to the dimension required with Type 1 Lyapunov functions. All of the increase comes from a different form of the adaptive gain needed to handle the unknown inertia matrix. Note that the adaptation law in the form of Eqs.(40,41) has been also developed for a different type of the attitude error.[8]

CONTROL LAW EXTENSIONS

The requirement imposed on both Type 1 and Type 2 Lyapunov functions to have negative definite derivatives in $\mathbf{s}$ presents a significant complication to MRAC development, which is explained in the following remarks.

Remark 7:

The problem is particularly difficult for Type 1 Lyapunov functions. At the present time, we are unaware of any controls, stability of which can be proven via Type 1 functions with derivative negative definite in $\mathbf{s}$. Since there is a number of control laws stability of which is established with "almost" Type 1 Lyapunov functions,[3-5] the performance of MRAC is now evaluated for these cases. "Almost" Type 1 Lyapunov functions have $\dot{V}_{01} \leq 0$, which is negative definite in w. They as well as the associated control laws follow naturally from the passivity properties of both the rigid-body dynamics and the rotational kinematics.[3,5] The asymptotic stability of the attitude itself is established via LaSalle's invariance principle.[3-5,13] The fact that the largest invariant set in $\left\{(\mathbf{s}, \mathrm{w}) : \mathrm{w} = \mathbf{0}\right\}$ implies $\mathbf{s} = \mathbf{1}_{\mathbf{s}}$ is established as the only solution to $\mathbf{u}(\mathbf{s},\mathbf{0})_{\mathbf{O}} = \mathbf{0}$. However, the adaptive operator $\tilde{\tilde{L}}_{r1}$ augments the original control, so that $\tilde{\tilde{L}}_r\mathbf{u}(\mathbf{s},\mathbf{0})_{\mathbf{O}} = \mathbf{0}$ no longer implies $\mathbf{s} = \mathbf{1}_{\mathbf{s}}$ unless the reference frame is inertial or the inertia matrix and the disturbance torque are estimated exactly. Since the estimates are not guaranteed to be exact, the MRAC extension only achieves tracking within a constant attitude offset. The following example illustrates this result.

Example 1:

Let the original control law be[4]

$$\mathbf{u}(\mathrm{s}, \mathrm{w})_{\mathbf{O}} = -k_1\mathrm{s} - k_2\mathrm{w}, \forall k_1, k_2 \in \Re^{+}, \qquad (45)$$

where s is the modified Rodrigues parameters[10] representation of $SO(3)$ used here in place of $\mathbf{s}$. Using $\tilde{\tilde{L}}_r\mathbf{u}(\mathrm{s},\mathbf{0})_{\mathbf{O}}$ even in the absence of uncontrolled torque leads to s asymptotically approaching a constant value, which is a solution of

$$\begin{aligned} k_1\mathrm{s} &= \mathrm{d}\mathbf{IC}(\mathrm{s})\dot{\overline{\mathrm{w}}}_{\mathbf{r}} \\ &+ \left[\mathbf{C}(\mathrm{s})\overline{\mathrm{w}}_{\mathbf{r}}\right]^{\times}\mathrm{d}\mathbf{IC}(\mathrm{s})\overline{\mathrm{w}}_{\mathbf{r}} \end{aligned} . \qquad (46)$$

Hence, $\mathrm{s} = \mathbf{0}$ if either $d\mathbf{I} \equiv \mathbf{0}$ or $\left(\overline{\mathrm{w}}_{\mathbf{r}}, \dot{\overline{\mathrm{w}}}_{\mathbf{r}}\right) \equiv (\mathbf{0,0})$, otherwise it may not be $\mathbf{0}$.

Remark 8:

The question may arise whether there are any controls satisfying Type 2 Lyapunov functions. The direct answer can be found via the application of the structured dynamic model inversion[8] proposed as a way of finding control laws that produce the prescribed error dynamics. This methodology first leads to a desired angular acceleration $\dot{\mathrm{w}}^{\mathbf{d}}$, which can in turn be used to solve for the required net torque. Selecting the asymptotically stable attitude error dynamics leads directly to the Type 2 Lyapunov function. In fact, the operator $\overline{L}_r$ can be easily modified to act on $\dot{\mathrm{w}}^{\mathbf{d}}$ itself.

Corollary 5:

The operator $\overline{L}'\left\{\Re^3 \times \Re^3\right\}: \Re^3 \to \Re^3$

$$\overline{L}_r' \overset{\Delta}{=} \overline{L}'\left\{\overline{\mathrm{w}}_{\mathbf{r}}, \dot{\overline{\mathrm{w}}}_{\mathbf{r}}\right\} \overset{\Delta}{=} \mathbf{I}E + \mathrm{W}^{\times}\mathbf{I}\mathrm{W} - \mathrm{w}^{\times}\mathbf{I}\mathrm{w} + \mathbf{I}\left[\dot{\overline{\mathrm{w}}}_{\mathbf{B}}\right]_{\mathbf{r}}, \forall \overline{\mathrm{w}}_{\mathbf{0r}}, \dot{\overline{\mathrm{w}}}_{\mathbf{r}} \in \Re^3 \quad (47)$$

acting on the desired angular acceleration $\dot{\mathrm{w}}^{\mathbf{d}}$ generates such net control torque that produces the prescribed error dynamics.

Note that the new operator is still linear in the inertia matrix, so the standard MRAC methodology is similarly applicable.

Example 2:

A simple choice for the asymptotically stable attitude error dynamics is a damped oscillator, which is presented below in the form of modified Rodrigues parameters:[10]

$$\ddot{\mathrm{s}} + c\dot{\mathrm{s}} + k\mathrm{s} = \mathbf{0}, \forall c, k \in \Re^+. \quad (48)$$

The resulting adaptive control has been formulated using a somewhat different definition of the attitude error:[8] $\mathrm{s} \overset{\Delta}{=} \mathrm{S} - \overline{\mathrm{s}}_{\mathbf{r}}$:, where S indicates the rigid-body attitude with respect to the inertial frame $\left\{\hat{\mathbf{i}}_{\mathbf{i}}\right\}$. A very similar derivation can be obtained simply by following the methodology described in this paper, which produces the Type 2 Lyapunov function control and its corresponding adaptation law.

As shown in this paper, Type 1 Lyapunov functions can lead to a much more efficient adaptation law. At the present time, it is unclear if such functions can be found via the structured dynamic model inversion.

The next extension presented in this paper augments the form of the original operator $\overline{L}_r$ when only the asymptotic stability needs to be preserved and not the exact error dynamics.

Remark 9:

The original operator $\overline{L}_r$ can be modified without affecting asymptotic stability provided the modifications appear in the tangent space of the Lyapunov function. In particular, this extension has been utilized without acknowledging its existence in order to generate velocity-free attitude tracking control law.[7] Since this is a very useful form of the extension, it is formally presented in the following corollary.

Corollary 6:

The following operator $\overline{L}''\left\{\Re^3 \times \Re^3\right\}: \Re^3 \to \Re^3$, which does not require knowledge of the angular velocity, preserves the asymptotic stability of the original control law provided that there exists a Lyapunov function $V(\mathbf{s}, \mathrm{w}) > 0$ such that $\dfrac{\partial V(\mathbf{s}, \mathrm{w})}{\partial \mathrm{w}} = [\mathbf{I}\mathrm{w}]^{\mathbf{T}}$:

$$\overline{L}_r'' \overset{\Delta}{=} \overline{L}''\left\{\overline{\mathrm{w}}_{\mathbf{r}}, \dot{\overline{\mathrm{w}}}_{\mathbf{r}}\right\} \overset{\Delta}{=} E + \mathbf{IC(s)}\dot{\overline{\mathrm{w}}}_{\mathbf{r}} + \left[\mathbf{C(s)}\overline{\mathrm{w}}_{\mathbf{r}}\right]^{\times} \mathbf{IC(s)}\overline{\mathrm{w}}_{\mathbf{r}}, \forall \overline{\mathrm{w}}_{\mathbf{r}}, \dot{\overline{\mathrm{w}}}_{\mathbf{r}} \in \Re^3. \quad (49)$$

This result is obtained by removing terms lying in the tangent space of $V(\mathbf{s}, \mathrm{w})$ and by using triple product identities to eliminate remaining dependencies on the angular velocity.

Example 3:

Several angular velocity free stabilizing controllers have been proposed recently.[3,5] Their development utilizes passivity of both the rigid-body dynamics and the rotational kinematics. The passivity of the former is easily established with the use of the well-known Lyapunov function: $V_w(\mathrm{w}) = \dfrac{1}{2}\mathrm{w}^{\mathbf{T}}\mathbf{I}\mathrm{w}$. The overall Lyapunov function is then created as the sum of $V_w(\mathrm{w})$ and another function, $V_s(\mathbf{s})$, of the particular attitude representation $\mathbf{s}$. Clearly, the resulting function satisfies the criterion of *Corollary 6*, thus, permitting application of the operator L_r''. This way the velocity-free tracking controller can be established for any of the existing velocity-free stabilizing controllers. In particular, one recently proposed tracking control law[7] follows directly from the simple form of the more general stabilizing controller.[5] Note that, while the stability of such tracking controllers can of course be proven, there is no need to do so if they are generated from the existing stabilizing controllers via the operator L_r''.

Finally, it should be mentioned that the adaptive re-design presented in this paper can be extended to the case of non-constant bounded disturbances. The re-design should follow methodology similar to the one developed for a different form of the attitude error.[6]

CONCLUSIONS

A family of trajectory dependent operators acting on the input torque is introduced. The basic operators mechanize mapping for preserving attitude motions with respect to different moving reference frames. As a result, duality of attitude stabilization and tracking is established. General dependency of tracking control laws on the inertia matrix is also demonstrated. Then the advanced operators are developed for adapting the mapping in the presence of the unknown inertia matrix and constant uncontrolled torque. The minimum order of the adaptive gains is evaluated for different types of control laws. Finally, various extensions to basic operators are presented, including coupling with the structured dynamic model inversion and angular velocity free controllers. All of the results presented are valid for any choice of the attitude representation.

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