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EFFECT OF COORDINATE SELECTION ON ORBIT DETERMINATION

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The application of linear estimation techniques to the non-linear problem of orbit determination requires that a linearization about the spacecraft trajectory be performed. This linearization process can be performed in any number of different coordinates. The effect of coordinate selection on the resulting orbit state and orbit state error covariance in batch and sequential estimators is investigated through examination of the mathematics of the estimators and a numerical example. Estimation in equinoctial elements is seen to be preferred to Cartesian coordinates in both batch and sequential estimation when the orbital motion is dominated by two-body dynamics. Means for affecting a change in the estimation coordinates through localized modifications to batch and sequential estimators are identified which allow for simple conversion of an estimator operating in Cartesian coordinates to be equivalent to an estimator operating in equinoctial elements.

INTRODUCTION

Traditional orbit estimation employs linear estimation techniques to estimate corrections to the non-linear trajectory of the satellite. As both the dynamical model and the measurement models are non-linear, the application of linear methods to the problem of orbit determination requires that linearization be performed about the nominal trajectory of the satellite and on the mathematical model of the measurements. The linearization about the trajectory may be global, as is the case in batch weighted least squares (BWLS) estimators, or it may be local, as is the case in the extended Kalman filter (EKF). In either case, the linearization of the trajectory and measurement processes allows the mapping of measurement residuals into state corrections via the linear machinery of the estimator. In this analysis, we are primarily concerned with the linearization of the trajectory.

The linearization process, which represents variations about the nominal trajectory using a first order Taylor series expansion, can be performed in any coordinates. Different coordinates are generally related by a non-linear transform, while variations in coordinates are related linearly by the appropriate Jacobian matrices. If the problem being treated is truly linear, there should be no preferential coordinates (all coordinates are equally valid) outside of issues of numerical computation. In the case of extremely small variations around the non-linear nominal orbit trajectory, the same conclusion is generally accepted. It has been suggested, however, that as variations from the nominal trajectory are allowed to grow in magnitude, certain coordinates yield superior linear-

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ization properties relative to other potential coordinate selections in the problem of orbit determination. Specifically, orbital elements have been stated to have superior linearization properties compared to Cartesian coordinates. This conclusion is reached based on superior agreement between the orbit error covariance function associated with errors about a non-linear nominal trajectory and populations of sample variations which have been integrated in a non-linear fashion[1,2,3,4,5]. It should be noted that there is little significance to the distinction between different coordinate selections for the case of determination of maintained orbits[2]. While coordinate preference in this sense has been demonstrated in multiple studies, the connection to the orbit determination process requires additional study.

The questions to be examined in this research are:

1. Is there a difference in the non-linear orbit determination solution achieved using different coordinates during the orbit determination process?
2. Is there a difference in the information content of the orbit error covariance function based on the use of different coordinates during the orbit determination process?
3. Where in the orbit determination algorithm is linearization applied and can the effect of coordinate selection be localized to these points?

The first question (i.e. the effect of coordinate selection on the orbit determination solution) is examined both through the mathematical machinery of the estimation method (batch weighted least squares and extended Kalman filter) and through numerical investigation. To facilitate the numerical study, an extension to AGI's Orbit Determination Tool Kit (ODTK)[6] has been constructed which allows for estimation to be performed natively in either Cartesian or equinoctial coordinates. A further augmentation to ODTK is made which allows for Cartesian coordinates to be included in the estimation state while state corrections are constructed in a manner which emulates estimation in equinoctial elements.

A difference in the information content of the orbit error covariance function exists if there is a difference, beyond that expected from numerical truncation errors, between the covariance obtained with two sets of coordinates after transformation to a common set of coordinates using the appropriate Jacobian matrices. We first examine the state error transition function, used in both BWLS and the EKF, for coordinate dependence. The state error transition function is used in BWLS to map measurement residuals to a common epoch and in the EKF during the time update of the state error covariance. We then investigate the non-linear state update process in each type of estimator to determine how and where coordinate selection affects the estimation process.

Finally, we examine the computational algorithms of the batch weighted least squares and extended Kalman filter estimators to determine if algorithmic changes based on coordinate selection can be localized to a small number of areas. Proper abstraction in the software design allows for the efficient implementation of new coordinates for research purposes or to satisfy requirements. Specifically, we show that the improved performance seen when estimating in equinoctial elements can be achieved through minor augmentations of the estimation algorithm when estimating in Cartesian coordinates.

GENERAL ASSUMPTIONS

A set of general assumptions are made which apply to both the batch weighed least squares and extended Kalman filter estimation methods for the purpose of this analysis. It is assumed that,

- propagation of the non-linear orbit trajectory can be performed in any coordinates with equivalent accuracy,

- evaluation of the non-linear observation model can be performed in any coordinates with equivalent accuracy,
- required non-linear transformations between coordinates can be computed to sufficient accuracy so as not to adversely affect the accuracy of the estimation process,
- required partial derivatives can be computed to sufficient accuracy so as not to adversely affect the accuracy of the estimation process.

The first two assumptions are easily satisfied if the third assumption of accurate non-linear coordinate transformations is satisfied since a single set of coordinates can be selected for the non-linear orbit and observation model computations. With regards to the fourth assumption, it is well known that certain coordinates have singularities which limit their range of application. For example, classical orbital elements have singularities at zero inclination and zero eccentricity. While the existence of such limitations is acknowledged, in this study we will limit ourselves to the use of equinoctial elements where such concerns are minimized.

In addition to the orbit state, force modeling parameters such as the ballistic coefficient and solar pressure coefficient are typically estimated. Since the definition of these parameters is independent of the coordinates used to represent the orbit state, we choose not to address them directly in this analysis. For the same reason, we do not address measurement model specific estimation states such as biases and stochastic clock parameters.

BACKGROUND

We define two sets of coordinates, Cartesian position and velocity and equinoctial elements represented in the same inertial reference frame[7,8]. The orbit state representations in Cartesian coordinates X and the equinoctial elements α are related by non-linear transformations.

$$X = G(\alpha) \, , \tag{1}$$

$$\alpha = U(X) \, . \tag{2}$$

A variation in one set of coordinates is mapped to a variation in another set of coordinates using the Jacobian between the two coordinates,

$$\Delta X = \frac{\partial X}{\partial \alpha} \Delta \alpha + \text{H.O.T.} \, , \tag{3}$$

and

$$\Delta \alpha = \frac{\partial \alpha}{\partial X} \Delta X + \text{H.O.T.} \, , \tag{4}$$

where H.O.T. denotes higher order terms than first order. The state error covariance associated with an estimate $\hat{X}$ of the true state X is defined as the expected value of the outer product of the state error[9],

$$P_X = E\left[\left(\hat{X} - X \right) \left(\hat{X} - X \right)^T \right] \, . \tag{5}$$

Defining $\Delta X = \hat{X} - X$, we can express P_X as

$$P_X = E\left[\Delta X\, \Delta X^T\right] + \text{H.O.T.}\,. \tag{6}$$

Substituting the relation between state differences in different coordinates gives

$$P_X = E\left[\frac{\partial X}{\partial \alpha} \Delta\alpha\, \Delta\alpha^T \frac{\partial X}{\partial \alpha}^T\right] + \text{H.O.T.}\,. \tag{7}$$

Since the Jacobian itself is not a random variable, it can be moved outside the expectation operator to arrive at

$$P_X = \frac{\partial X}{\partial \alpha} E\left[\Delta\alpha\, \Delta\alpha^T\right] \frac{\partial X}{\partial \alpha}^T + \text{H.O.T.}\,. \tag{8}$$

The expectation operation on the outer product of errors in the orbit elements yields the state error covariance expressed in terms of the orbit elements. The resulting expression provides the transformation of state error covariance between different coordinates[10],

$$P_X = \frac{\partial X}{\partial \alpha} P_\alpha \frac{\partial X}{\partial \alpha}^T \quad . \tag{9}$$

The relationship given in Equation (9) describes an equivalence of information content in the covariance matrix. The previously referenced studies have shown that error covariance expressed in certain coordinates provides a better model of the error distribution than when it is expressed in other coordinates. We will use Equation (9) in the sequel to determine if a change in the coordinates used for estimation results in a change in the information content as expressed in the state error covariance.

The observation models in BWLS and EKF estimators are non-linear functions of the orbit state. Let Y_{Obs} represent measured observations and Y represent the modeled values of those observations. The observation residual y is defined as the difference between the measured and modeled values of the observations,

$$y = \Delta Y = Y_{Obs} - Y \ . \tag{10}$$

The non-linear model of the observations is computed using the non-linear representation of the trajectory. Since the non-linear trajectory is the same in different coordinates, the modeled value of an observation has no dependence on the coordination selection for the estimator. The measurement residual is therefore also independent of the coordination selection.

The Jacobian of the measurement representation with respect to the estimation state is required to map measurement residuals into state corrections. The measurement Jacobian matrices can be written as

$$H_X = \frac{\partial Y}{\partial X} \tag{11}$$

for the case of Cartesian coordinates and

$$H_\alpha = \frac{\partial Y}{\partial \alpha} \tag{12}$$

when equinoctial elements are used to represent the trajectory. H_X and H_α are related by the Jacobian between the two sets of coordinates,

$$H_X = \frac{\partial Y}{\partial \alpha}\frac{\partial \alpha}{\partial X} = H_\alpha \frac{\partial \alpha}{\partial X} \ . \tag{13}$$

Both the BWLS and EKF estimators use linearized time transition of variations about the state and linear measurement updates. The state error transition matrix Φ is used in the BWLS estimator to map state differences at the solution epoch t_0 to state differences at each measurement time t. In the EKF estimator, the state error transition matrix is used to move the state error covariance forward in time between measurements. The transition of state differences represented in Cartesian coordinates and equinoctial elements are represented as[11],

$$\Delta X(t) = \Phi_X(t_0, t)\Delta X(t_0) \ , \tag{14}$$

$$\Delta \alpha(t) = \Phi_\alpha(t_0, t)\Delta \alpha(t_0) \ . \tag{15}$$

Substituting Equation (4) into Equation (14) yields

$$\frac{\partial \alpha}{\partial X}\Delta X(t) = \Phi_\alpha(t_0, t)\frac{\partial \alpha(t_0)}{\partial X(t_0)}\Delta X(t_0) \ , \tag{16}$$

and using $\left(\frac{\partial \alpha}{\partial X}\right)^{-1} = \frac{\partial X}{\partial \alpha}$ we find

$$\Delta X(t) = \frac{\partial X}{\partial \alpha}\Phi_\alpha(t_0, t)\frac{\partial \alpha(t_0)}{\partial X(t_0)}\Delta X(t_0) \ . \tag{17}$$

Comparing Equation (14) and Equation (17) we see that[11]

$$\Phi_X(t_0, t) = \frac{\partial X}{\partial \alpha}\Phi_\alpha(t_0, t)\frac{\partial \alpha(t_0)}{\partial X(t_0)} \ . \tag{18}$$

The inverse relationship to compute the state error transition matrix in equinoctial elements from the state error transition matrix in Cartesian coordinates is given as

$$\Phi_\alpha(t_0, t) = \frac{\partial \alpha}{\partial X}\Phi_X(t_0, t)\frac{\partial X(t_0)}{\partial \alpha(t_0)} \ . \tag{19}$$

BATCH WEIGHTED LEAST SQUARES ESTIMATION

In BWLS estimation, a global linearization is used to enable the application of the linear algorithm to the non-linear problem of orbit determination. In using global linearization, a single non-linear reference trajectory serves as the reference about which variations are represented. The non-linearity of the problem is addressed through the use of multiple iterations of linear corrections at the solution epoch until the process has converged. With each linear correction, the non-linear reference solution is updated to serve as a new reference for the next iteration.

Minimization of the sum of squares of residuals

The BWLS algorithm is derived with the goal of minimizing the sum of the squares of the weighted measurement residuals. Based on the fact that the measurement residuals are independent of the coordinate selection, we can state that the non-linear trajectory which minimizes the BWLS cost function is also independent of the selection of coordinates used for linearization. This statement is based on the presumption of convergence to the same local minimum.

The path to convergence

Though the actual non-linear state which minimizes the sum of squares of the residuals is independent of coordinates, the path taken from the initial conditions to the solution may vary with the selection of coordinates. As mentioned, the BWLS algorithm overcomes the non-linear nature of the orbit determination problem through iteration. The mapping of the measurement partials to the BWLS epoch requires the state error transition matrix which represents the linearized dynamics in a particular set of coordinates. For observation i, this mapping is represented as

$$H_{X,i} = \frac{\partial Y_i}{\partial X(t_0)} = \frac{\partial Y_i}{\partial X} \Phi_X(t_0, t_i) \tag{20}$$

in Cartesian coordinates and

$$H_{\alpha,i} = \frac{\partial Y_i}{\partial \alpha(t_0)} = \frac{\partial Y_i}{\partial \alpha} \Phi_\alpha(t_0, t_i) \tag{21}$$

in equinoctial elements. The normal equation, expressed here in Cartesian coordinates[9],

$$\Delta X = \left(H_X^T W H_X\right)^{-1} H_X^T W y \ , \tag{22}$$

where y is an array of residuals and W is the weighting matrix, provides one possible method of computing the update to the non-linear state during the BWLS iteration. We can use the normal equation to demonstrate the effect of coordinate selection on the BWLS solution path. From Equation (13) and Equation (20), we see that

$$H_{X,i} = \frac{\partial Y_i}{\partial X(t_0)} = \frac{\partial Y_i}{\partial \alpha(t_0)} \frac{\partial \alpha(t_0)}{\partial X(t_0)} = H_{\alpha,i} \frac{\partial \alpha(t_0)}{\partial X(t_0)} \ . \tag{23}$$

Substituting Equation (23) into the normal equation, Equation (22), gives

$$\Delta X = \left(\left(\frac{\partial \alpha(t_0)}{\partial X(t_0)}\right)^T H_\alpha^T W H_\alpha \frac{\partial \alpha(t_0)}{\partial X(t_0)} \right)^{-1} \left(\frac{\partial \alpha(t_0)}{\partial X(t_0)}\right)^T H_\alpha^T W y \ . \tag{24}$$

By inspection we note that the inverse in Equation (24) is given by

$$\left(\frac{\partial\alpha(t_0)}{\partial X(t_0)}^T H_\alpha^T W H_\alpha \frac{\partial\alpha(t_0)}{\partial X(t_0)}\right)^{-1} = \frac{\partial\alpha(t_0)}{\partial X(t_0)}^{-1} \left(H_\alpha^T W H_\alpha\right)^{-1} \frac{\partial\alpha(t_0)}{\partial X(t_0)}^{-T} . \tag{25}$$

If the residual weighting is chosen as the inverse of the variances of the measurement noise, then the least squares covariance is given as[11],

$$P_x = \left(H_X^T W H_X\right)^{-1} . \tag{26}$$

for Cartesian coordinates and

$$P_\alpha = \left(H_\alpha^T W H_\alpha\right)^{-1} \tag{27}$$

for equinoctial elements. Examination of Equation (25) then provides the relationship between the least squares covariance in Cartesian elements and the least squares covariance in equinoctial elements,

$$P_x = \frac{\partial\alpha(t_0)}{\partial X(t_0)}^{-1} P_\alpha \frac{\partial\alpha(t_0)}{\partial X(t_0)}^{-T} . \tag{28}$$

Noting that $\left(\frac{\partial\alpha}{\partial X}\right)^{-1} = \frac{\partial X}{\partial\alpha}$, we see that Equation (28) is equivalent, as expected, to Equation (9). Substitution of Equation (25) into Equation (24) gives

$$\Delta X = \frac{\partial X(t_0)}{\partial\alpha(t_0)}\left(H_\alpha^T W H_\alpha\right)^{-1} H_\alpha^T W y . \tag{29}$$

We recognize the existence of the normal equation solution expressed in equinoctial elements in the right hand side of Equation (29). Therefore, for a particular iteration, the BWLS correction in Cartesian coordinates is related to the correction in equinoctial elements simply by the Jacobian between the coordinate types,

$$\Delta X = \frac{\partial X(t_0)}{\partial\alpha(t_0)}\Delta\alpha , \tag{30}$$

which is, of course, in agreement with Equation (3). The final step in the BWLS iteration is to update the non-linear state, which is expressed as

$$X = X + \Delta X \tag{31}$$

for the case of Cartesian coordinates and

$$\alpha = \alpha + \Delta\alpha \tag{32}$$

for equinoctial elements. We note that since higher order terms exist in the transformation between Cartesian coordinates and equinoctial elements,

$$X + \Delta X \neq G(\alpha + \Delta\alpha) \ . \tag{33}$$

Thus despite the fact that the iteration process will eventually lead both solutions to the same result, within the limits of BWLS convergence, the path to achieving the solution will be different. We also note that Equation (30) indicates that the coordinates in which the BWLS estimator operates can be changed through a simple modification to the non-linear state update. For example, to convert a BWLS estimator operating in Cartesian coordinates to perform like one operating in equinoctial elements, the Cartesian state update is converted to an equinoctial state update using Equation (30). The non-linear state is then updated in equinoctial elements and transformed back to Cartesian coordinates using the non-linear coordinate transformation described by Equation (1).

The dependence of the convergence path upon coordinate selection means that it is possible for the number of iterations required for the BWLS algorithm to converge to depend on the coordinate selection. There may also be cases where convergence is achieved using one set of coordinates while the use of a different set of coordinates leads to divergence of the algorithm. Such cases would indicate a difference in the capture region of the estimator. If measurement editing is being employed, editing decisions on particular iterations can also differ. Disparate editing decisions on the final iteration would result in a different state solution.

SEQUENTIAL ESTIMATION

In the EKF, local linearization is employed to enable the application of the linear Kalman filter algorithm to non-linear problems such as orbit determination. Each time a new observation is processed, the non-linear trajectory is updated and serves as the new reference for linearization. The EKF is not iterated by design, but does typically experience an initialization period due to a lack of proper a priori covariance information. During this initialization period, the covariance structure develops based on the dynamics of the problem and the processing of measurements. The EKF is more sensitive to divergence during the initialization period leading to the conclusion that the EKF has a smaller capture region than BWLS. At the end of the initialization period, the filter is considered to be converged or steady-state and the filter covariance enables a powerful data editing capability. The EKF is a recursive machine of time and measurement updates. We will examine both types of updates to identify where coordinate selection plays a role.

Time Updates

The sequential time update algorithm moves the non-linear state and the associated state error covariance from the time of the time of the last observation (or the initial time) to the time of the next observation. As mentioned above, the transition of the non-linear state is assumed to be of equivalent accuracy, regardless of coordinate selection. The transition of the state error covariance is performed in a linear manner using the state error transition function about the nominal non-linear trajectory[11],

$$P_{X\,k+1|k} = \Phi_X(t_k, t_{k+1}) P_{X\,k|k} \Phi_X^T(t_k, t_{k+1}) + Q_X(t_k, t_{k+1}) \ , \tag{34}$$

where the notation $P_{k+1|k}$ indicates that the covariance represents uncertainty at time t_{k+1} including measurement information through time t_k and $Q_X(t_k, t_{k+1})$ is the additive process noise matrix which accounts for dynamical uncertainty over the interval from time t_k to time t_{k+1} and is added at time t_{k+1}. Substituting Equation (18) and Equation (9) into Equation (34), applying a linear transformation to the process noise matrix $Q_X(t_k, t_{k+1})$ and simplifying yields

$$P_{X\,k+1|k} = \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})} \Phi_\alpha(t_k, t_{k+1}) P_{\alpha\,k|k} \Phi_\alpha^T(t_k, t_{k+1}) \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})}^T + \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})} Q_\alpha(t_k, t_{k+1}) \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})}^T , \tag{35}$$

which simplifies to

$$P_{X\,k+1|k} = \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})} \left[\Phi_\alpha(t_k, t_{k+1}) P_{\alpha\,k|k} \Phi_\alpha^T(t_k, t_{k+1}) + Q_\alpha(t_k, t_{k+1}) \right] \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})}^T . \tag{36}$$

Recognizing that the term inside the brackets is simply $P_{\alpha\,k+1|k}$ allows us to write

$$P_{X\,k+1|k} = \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})} P_{\alpha\,k+1|k} \frac{\partial X(t_{k+1})}{\partial \alpha(t_{k+1})}^T . \tag{37}$$

Equation (37) is again identical to Equation (9) and shows that the EKF time update algorithm results in the same information content for the state error covariance in all coordinates, based on the assumption that the non-linear reference trajectory is identical.

Measurement Updates

The sequential measurement update algorithm computes a correction to the state and an update to the state error covariance at the current time based on a measurement residual at the current time. The measurement updates of the state and the state error covariance using different coordinates can be shown to be related through the Jacobian relating the two sets of coordinates.

The state update for a measurement at time t_{k+1} for the Cartesian representation is expressed as[11]

$$X_{k+1|k+1} = X_{k+1|k} + K_X y_k . \tag{38}$$

where K_X is the Kalman gain defined as[11]

$$K_X = P_X H_X^T \left(H_X P_X H_X^T + R \right)^{-1} \tag{39}$$

and R is the measurement white noise variance which is independent of coordinate selection by definition. The part of K_X shown inside parentheses above is the measurement residual error variance. Using Equation (13) we see that,

$$H_X P_X H_X^T = H_\alpha \frac{\partial \alpha}{\partial X} P_X \frac{\partial \alpha}{\partial X}^T H_\alpha^T , \tag{40}$$

which can be simplified as

$$H_X P_X H_X^T = H_\alpha P_\alpha H_\alpha^T \tag{41}$$

to show the coordinate independence of the measurement residual error variance. We substitute Equation (9) and Equation (13) into the first two terms of Equation (39) to yield

$$K_X = \frac{\partial X}{\partial \alpha} P_\alpha \frac{\partial X}{\partial \alpha}^T \left(\frac{\partial \alpha}{\partial X}^T H_\alpha^T \right) \left(HPH^T + R \right)^{-1}, \tag{42}$$

where we have dropped the coordinate designation on the measurement residual error variance to reflect its coordinate independence. Equation (42) can be simplified to yield the relationship between the Kalman gain in Cartesian coordinates and the Kalman gain in equinoctial elements,

$$K_X = \frac{\partial X}{\partial \alpha} P_\alpha H_\alpha^T \left(HPH^T + R \right)^{-1} = \frac{\partial X}{\partial \alpha} K_\alpha . \tag{43}$$

The relationship between the gains can be applied to Equation (38) to provide the relationship between the measurement state update in the different coordinates,

$$\Delta X = K_X y = \frac{\partial X}{\partial \alpha} K_\alpha y = \frac{\partial X}{\partial \alpha} \Delta \alpha , \tag{44}$$

or in the opposite sense,

$$\Delta \alpha = K_\alpha y = \frac{\partial \alpha}{\partial X} K_X y = \frac{\partial \alpha}{\partial X} \Delta X . \tag{45}$$

Equations (44)-(45) indicate that coordinate selection in the measurement state update can be implemented by computing the modeled measurements and measurement partial derivatives in a single set of coordinates and then transforming the state update to the coordinates native to the estimation process. For example, the measurement state update for an augmented EKF which operates in Cartesian coordinates but emulates an EKF operating in equinoctial elements is accomplished by computing the normal update to the Cartesian state update

$$\Delta X = K_X y , \tag{46}$$

then transforming the result to an update in equinoctial elements,

$$\Delta \alpha = \frac{\partial \alpha}{\partial X} \Delta X . \tag{47}$$

The non-linear state is updated in equinoctial elements,

$$\alpha_{k+1|k+1} = \alpha_{k+1|k} + \Delta \alpha \ . \tag{48}$$

Finally, the updated equinoctial elements are transformed back to Cartesian coordinates using the full non-linear transformation,

$$X_{k+1|k+1} = G\left(\alpha_{k+1|k+1} \right) . \tag{49}$$

The measurement covariance update in Cartesian coordinates is given by[11]

$$P_{X\,k+1|k+1} = P_{X\,k+1|k} - K_X H_X P_{X\,k+1|k} \ . \tag{50}$$

Substituting expressions in terms of equinoctial elements on the right hand side of Equation (50) yields

$$P_{X\,k+1|k+1} = \frac{\partial X}{\partial \alpha} P_{\alpha\,k+1|k} \frac{\partial X}{\partial \alpha}^T - \frac{\partial X}{\partial \alpha} K_\alpha H_\alpha P_{\alpha\,k+1|k} \frac{\partial X}{\partial \alpha}^T \ , \tag{51}$$

which further simplifies to

$$P_{X\,k+1|k+1} = \frac{\partial X}{\partial \alpha} P_{\alpha\,k+1|k+1} \frac{\partial X}{\partial \alpha}^T \ , \tag{52}$$

or in the opposite sense,

$$P_{\alpha\,k+1|k+1} = \frac{\partial \alpha}{\partial X} P_{X\,k+1|k+1} \frac{\partial \alpha}{\partial X}^T \ . \tag{53}$$

We note that the coordinate Jacobian matrices in Equations (52)-(53) are evaluated at time t_{k+1}. Equations (52) and (53) are the result that we expected, with one subtle distinction: the Jacobian matrices given in the right hand side of both equations are evaluated using the non-linear state prior to the measurement update of the state.

Let us now consider the case where we wish to augment an EKF operating in one set of coordinates to perform the measurement update in another set of coordinates (i.e. changing an estimator operating in Cartesian coordinates to emulate estimation in equinoctial coordinates). In this circumstance, we need to remember that the a posteriori covariance is associated with the a posteriori state. The covariance measurement update augmentation to emulate estimation in equinoctial elements therefore starts with the computation of the normal covariance update in Cartesian coordinates given by Equation (50). The Cartesian representation of the covariance $P_{X\,k+1|k+1}$ is then converted to equinoctial elements with the Jacobian computed using the a priori state $X_{k+1|k}$ (state at time t_{k+1} prior to the application of the measurement at time t_{k+1}),

$$P_{\alpha\,k+1|k+1} = \left[\frac{\partial \alpha}{\partial X}\right]_{k+1|k} P_{X\,k+1|k+1} \left[\frac{\partial \alpha}{\partial X}\right]_{k+1|k}^T \ . \tag{54}$$

The equinoctial representation of the state error covariance given in Equation (54) is the same as what would have been computed in an EKF operating natively in equinoctial elements. This state error covariance is a measure of the uncertainty in the a posteriori equinoctial state. To generate the equivalent a posteriori state error covariance in Cartesian elements, we need convert back to Cartesian coordinates using the Jacobian computed using the a posteriori state $X_{k+1|k+1}$ (state at time t_{k+1} after the application of the measurement at time t_{k+1}),

$$P_{X\,k+1|k+1} = \left[\frac{\partial X}{\partial \alpha}\right]_{k+1|k+1} P_{\alpha\,k+1|k+1} \left[\frac{\partial X}{\partial \alpha}\right]_{k+1|k1}^T . \tag{55}$$

COORDINATE PREFERENCE

We have shown that coordinate selection can affect the behavior of both BWLS and EKF estimators in the orbit determination problem. The question now becomes: Should we expect one set of coordinates to be preferred over another set? As previously mentioned, a number of studies have shown that covariance represented in orbital elements provides a better representation than Cartesian coordinates of the distribution of a set of trajectories which represent perturbations drawn from an initial Gaussian distribution about a nominal trajectory. This indicates that the linearization of the dynamics in terms of orbital elements has superior properties relative to linearization in terms of Cartesian coordinates.

In the two body problem, which dominates the overall dynamics in most cases, the particular choice of orbital elements where the energy is represented through the mean motion n and the angle in the orbit is given by the mean longitude L results in the equations of motion being linear rather than non-linear. Let the equinoctial elements be represented as

$$\alpha = \begin{bmatrix} a_f & a_g & n & L & \chi & \psi \end{bmatrix}^T = \begin{bmatrix} k & h & n & L & p & q \end{bmatrix}^T \tag{56}$$

where we use the mean motion as the energy parameter of the element set as originally expressed by Arsenault and Koskela[8] in preference to the semi-major axis length used by Broucke and Celfola[7] to take advantage of the improved linearization of the dynamics when using mean motion. The time derivative of the equinoctial elements under two body motion is given as

$$\dot{\alpha} = \begin{bmatrix} 0 & 0 & 0 & n & 0 & 0 \end{bmatrix}^T . \tag{57}$$

The time derivative of the linearized dynamics is given as

$$\Delta\dot{\alpha} = \begin{bmatrix} 0 & 0 & 0 & \Delta n & 0 & 0 \end{bmatrix}^T , \tag{58}$$

which leads to the following result

$$\dot{\alpha} + \Delta\dot{\alpha} = \frac{d}{dt}\left(\alpha + \Delta\alpha\right). \tag{59}$$

Since the two body dynamics are linear when expressed in these coordinates, the linearization of the dynamics is not an approximation of motion about the nominal trajectory, it is an exact representation of motion about the nominal trajectory. A Gaussian distribution will therefore propagate exactly as a Gaussian distribution for all time. In contrast, the two body dynamics expressed in Cartesian coordinates the two body dynamics are not linear. The linearized dynamics in Cartesian coordinates are an approximation valid only in a local region about the nominal. Thus the equinoctial set of coordinates should be preferred over Cartesian as their update is not an approximation.

The connection to orbit estimation comes in the update of the non-linear state. The state correction is computed, for both the BWLS and the EKF, under the assumption that the updated non-linear state will behave in a manner predicted by the linearized dynamics. This assumption will obviously be more valid for cases where the linearized dynamics are a better representation of the non-linear dynamics. In the context of this study, we therefore expect an update in equinoctial elements to demonstrate superior performance to an update in Cartesian coordinates. The significance of the improvement will be determined by size of the update. When updates are small, as is the case for the vast majority of orbit determination, empirical evidence demonstrates that either representation is adequate. We will therefore examine cases which are more representative of low resolution tracking with low data density to provide examples of the effect of coordinate selection on the orbit determination process.

NUMERICAL EXAMPLE

The following simplified example is chosen to demonstrate the effect of coordinate selection. A satellite in a high LEO orbit is tracked by a single tracking station using range, Doppler, azimuth and elevation measurements with a time between observations of 15 minutes. Approximate initial conditions for the orbit and initial error root variances on the position and velocity are given in Table 1. The error root variances are provided in the Gaussian frame (radial, along track, cross track). The initial conditions and tracking schedule for this test case is designed to produce a large initial uncertainty prior to the processing of measurements.

Table 1. Example case parameters.

Orbital Element	Value	Uncertainty	Value
a	8674 Km	σ_R	4768 m
e	0.003	σ_I	12026 m
i	68.7 deg	σ_C	7791 m
Ω	0.3 deg	σ_{RDot}	7.52 m/s
ω	207.5 deg	σ_{IDot}	3.79 m/s
ν	152.4 deg	σ_{CDot}	8.12 m/s

The initial covariance was generated by processing a small amount of data using a BWLS estimator. To test the equivalence of the state error transition function between the Cartesian and equinoctial element coordinate selections, the initial state error covariance was propagated using two body dynamics without process noise in both Cartesian coordinates and equinoctial elements. The state error covariance evolution was used as a proxy for the state error transition matrix since it is an existing output of ODTK. The state error covariance matrices were then transformed to the Gaussian frame for comparison. The resulting time histories of the one sigma position uncertainties are shown in Figures 1 and 2.

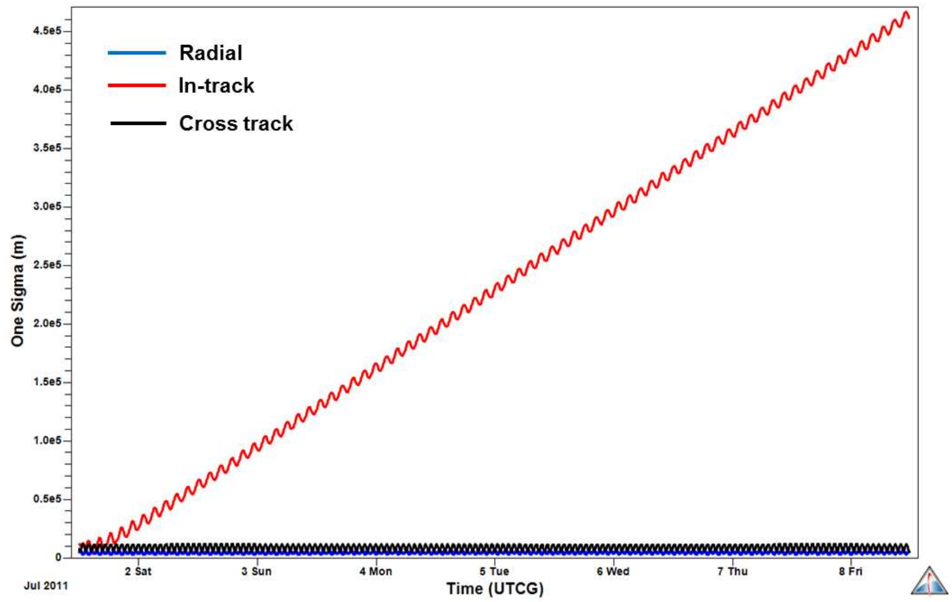

Figure 1. Position error covariance propagated in Cartesian coordinates.

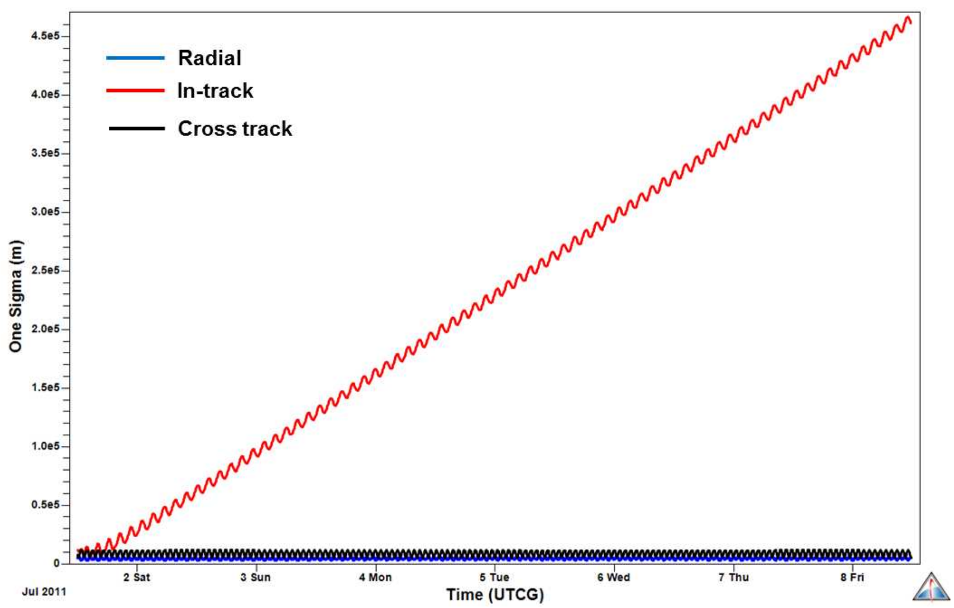

Figure 2. Position error covariance propagated in equinoctial elements.

It is interesting to note that despite the fact that the linearization in equinoctial elements is exact while the linearization in Cartesian coordinates is not, there is no difference in information content in the state error covariance histories when the covariance is propagated in a linear manner.

For testing of the estimation algorithms, the perturbation due to J_2 was added to the dynamical model. To push the estimation algorithms toward the computation large corrections, no data was processed for 3 days after the initial conditions. Figure 2 shows the results of the BWLS orbit determination. The number of iterations shown along the X axis includes the total number of iterations required to converge. Note that while both formulations start with the same initial weighted RMS (73970) and achieve the same final weighted RMS (0.9163), the number of iterations required using Cartesian coordinates is double the number required using equinoctial elements. Additionally, the convergence path in Cartesian coordinates includes one iteration where the weighted RMS increased while the convergence path in equinoctial elements contained no iterations where the RMS increased. The weighted RMS is the root mean square of the measurement residuals where each residual has been weighted by the inverse of the measurement noise root variance.

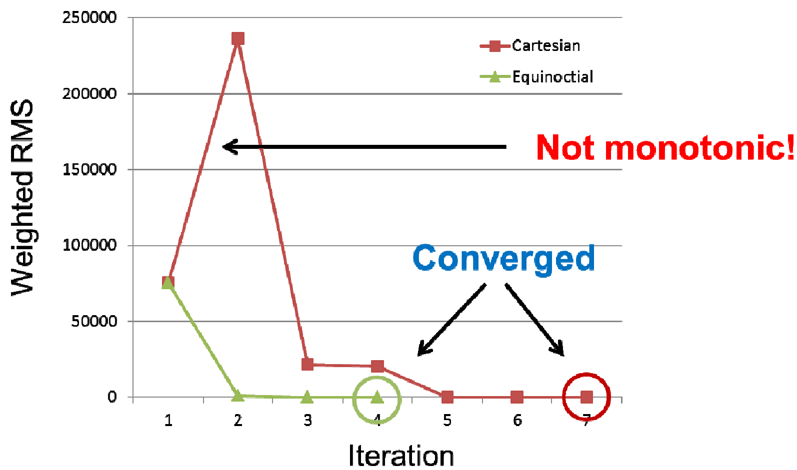

Figure 3. BWLS convergence using Cartesian coordinates and equinoctial elements.

Three variants of the EKF were tested: Estimating natively in Cartesian coordinates, estimating natively in equinoctial elements and estimating in Cartesian coordinates with an augmentation to perform the measurement update in equinoctial elements. Range and Doppler measurement residuals were computed during estimation with the EKF but were not accepted for measurement updates to avoid non-linear effects of the measurement model during the processing of the first measurements. Figures 4-5 contain the resulting residual ratio plots and position error covariance in the Gaussian frame for estimating natively in equinoctial elements and estimating in Cartesian coordinates. The case of estimating in Cartesian coordinates with an augmentation to perform the measurement update in equinoctial elements is indistinguishable from the case of estimating natively in equinoctial elements. The residual ratio plots show the time history of processed measurement residuals which have been divided by the corresponding measurement error root variance. Ideally, 99% of the residual ratios would lie between +/- 3. A trend of residual ratios moving away from the desired range of +/- 3 indicates divergence of the filter. As seen in Figure 3, the EKF in Cartesian mode diverged from the true trajectory indicating that the measurement update in Cartesian coordinates was too large to be within the linear range of the estimator.

Figures 6-7 contain plots of the position error covariance. Again, the case of estimating in Cartesian coordinates with an augmentation to perform the measurement update in equinoctial elements is indistinguishable from the case of estimating natively in equinoctial elements. The secular in-track error growth at the left side of both plots is due to propagation in the absence of measurements. At the time when the first measurements are processed, the in-track uncertainty is approximately 200Km (one sigma). The secular in-track error growth at the right side of Figure 5 is indicative of filter divergence.

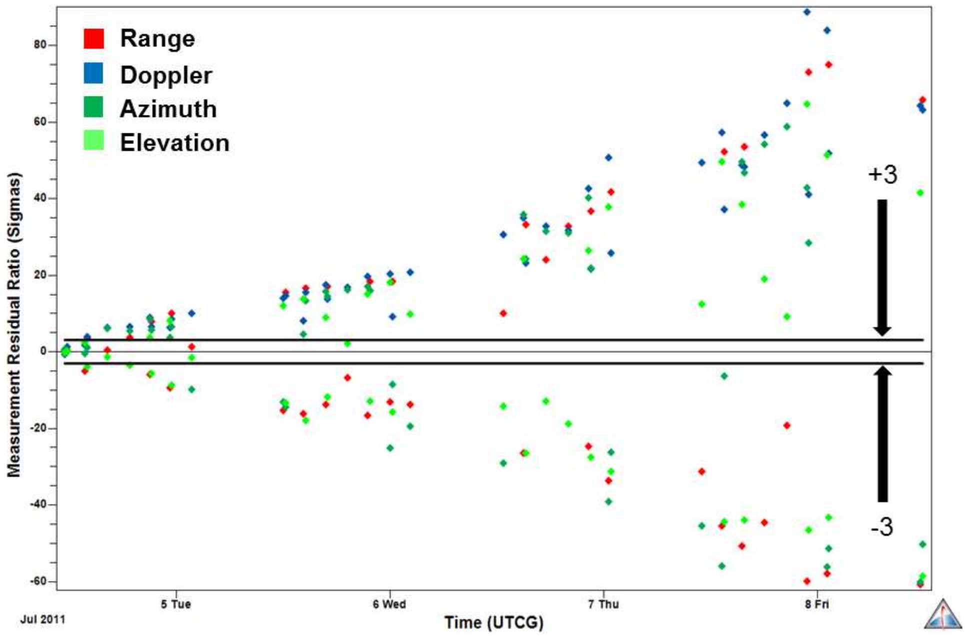

Figure 4. Residual ratios resulting from estimation in Cartesian coordinates.

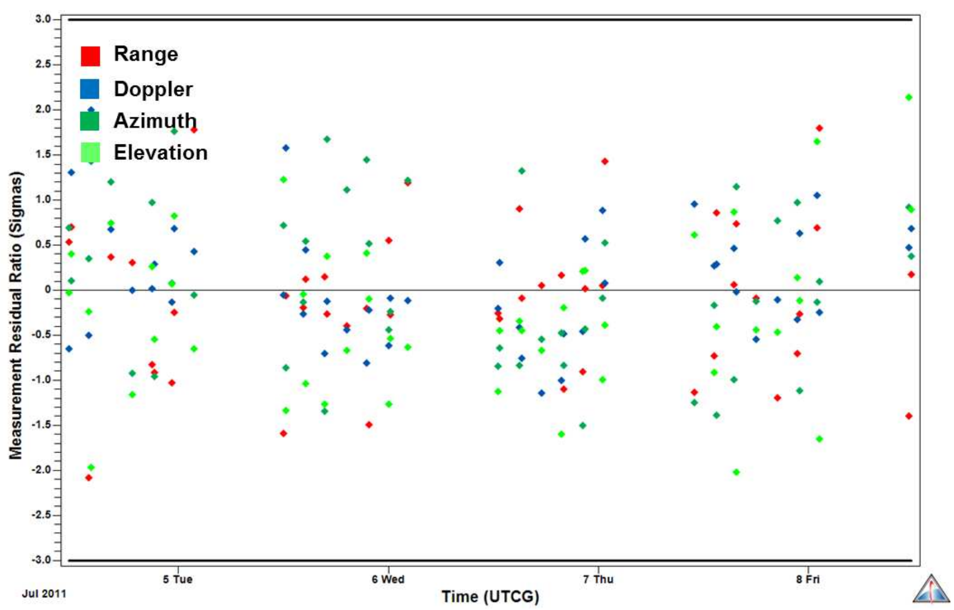

Figure 5. Residual ratios resulting from estimation in equinoctial elements and the Cartesian coordinates with measurement update in equinoctial elements.

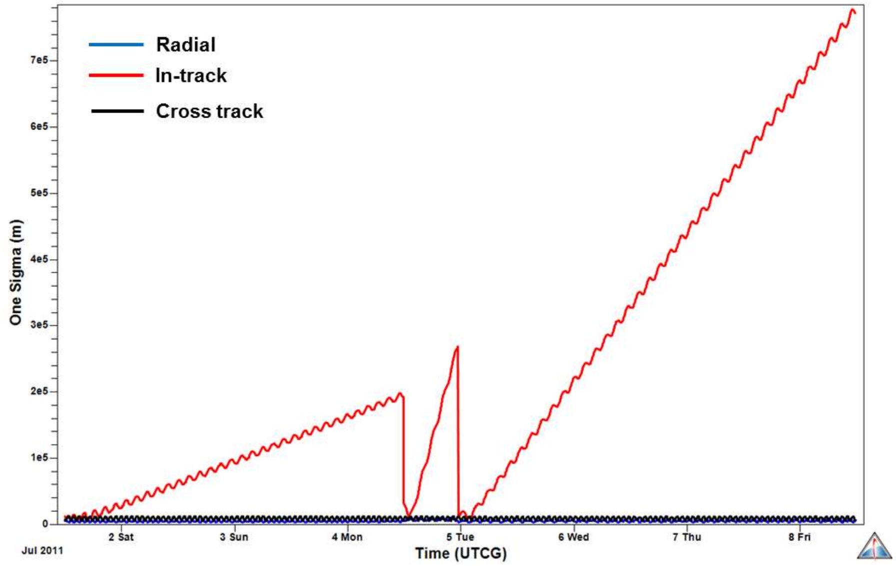

Figure 6. Position error covariance resulting from estimation in Cartesian coordinates.

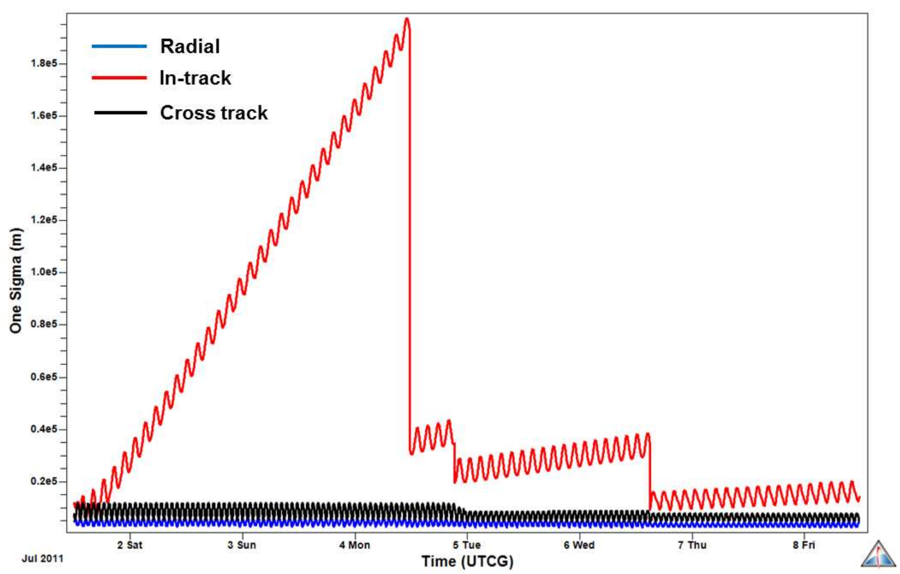

Figure 7. Position error covariance resulting from estimation in equinoctial elements and the Cartesian coordinates with measurement update in equinoctial elements.

Even prior to the divergence of the EKF, Figure 8, the state estimate is different from that produced by processing in equinoctial elements due to non-equivalence of the measurement update. These state differences then result in differences in the state error covariance. On the other hand,

Figure 9, the native equinoctial and the augmented Cartesian results are identical for all practical purposes (less than 1 cm of difference after 7 days).

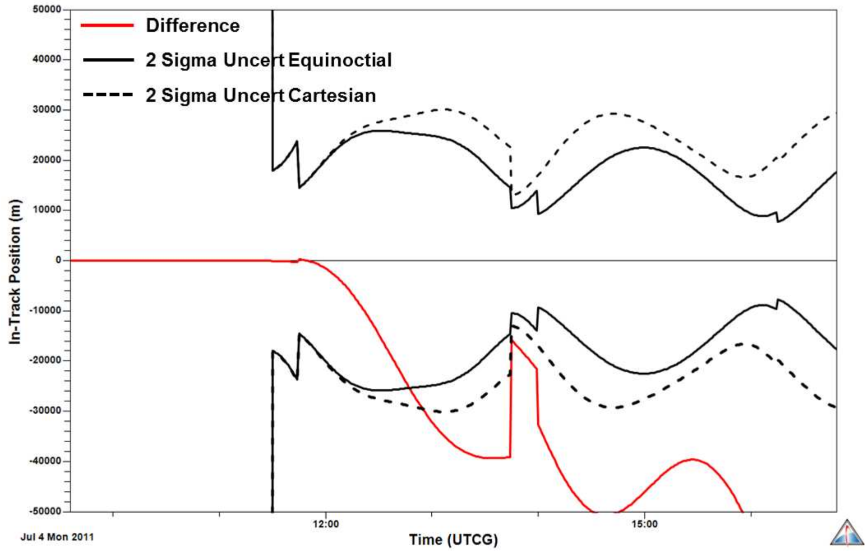

Figure 8. In-track position difference between Cartesian and equinoctial solutions and in-track position uncertainty from both solutions.

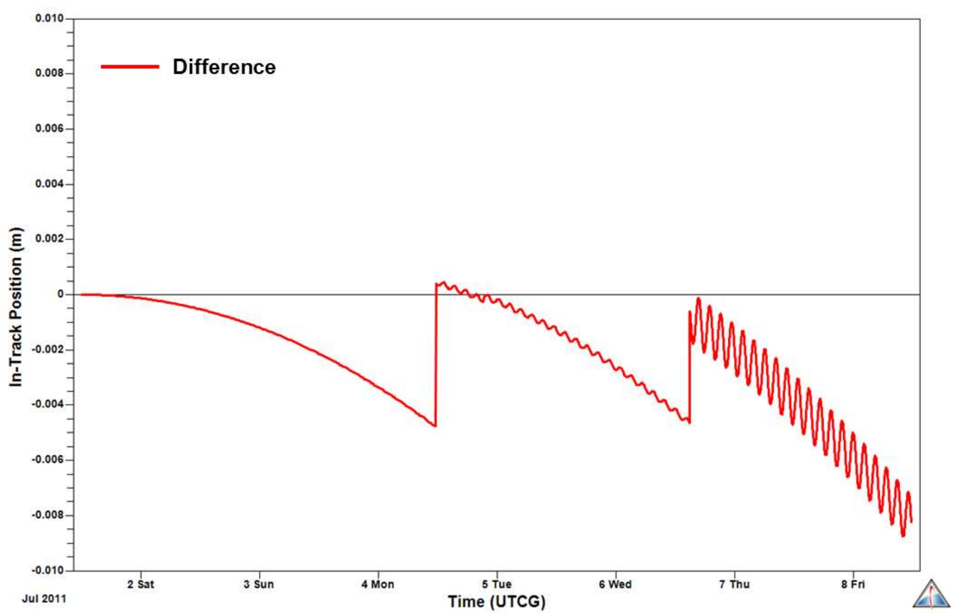

Figure 9. In-track position difference between native equinoctial and augmented Cartesian solutions.

CONCLUSION

BWLS and EKF estimation methods are modified linear estimation techniques commonly applied to the non-linear problem of orbit determination. Differences in the effect of coordinate selection between BWLS and EKF based orbit determination can be traced to a difference in linearization approaches.

In BWLS based estimation, the non-linear solution and associated error covariance which minimizes the sum of squares of the residuals is seen to be independent of the coordinate selection. Coordinate selection in BWLS orbit determination does affect the path taken to the final solution. This difference in the convergence path can lead to a difference in the number of iterations required for convergence or even to a difference in the capture region of the orbit determination process. A change in coordinate selection can be implemented through augmentation of the non-linear state update at the end of each BWLS iteration without modification of rest of the BWLS implementation.

In EKF based estimation, the non-linear solution and associated state error covariance are both influenced by the selection of trajectory coordinates. The coordinate dependence of the EKF algorithm results from the use of local linearization where the non-linear state is updated as each measurement is processed. Applying the linear measurement update in different coordinates results in a slightly different non-linear state and state error covariance. A change in coordinate selection can be implemented through augmentation of the measurement update algorithm without modification of the time update algorithm.

The use Cartesian coordinates in orbit determination is a widely accepted and successful practice. For the case where satellite motion is dominated by the two body dynamics, the selection of orbital elements, especially those containing the mean motion as the energy parameter, has been seen to provide the desirable characteristic that the dynamics are nearly linear (or exactly linear in the case of the two body problem). When state corrections are large, the selection of orbital elements as the estimation coordinates is seen to have superior convergence properties relative to Cartesian coordinates. The improved convergence properties are evident in both batch and sequential estimators and include a likely expansion of the capture region of the estimator.

The propagation of the post-fit state error covariance is seen to yield equivalent information content when performed in different coordinates. For example, the state error covariance propagated in Cartesian coordinates can be linearly transformed to yield the same result as the state error covariance propagated in equinoctial elements.

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