ATTITUDE COVARIANCE VISUALIZATION

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ABSTRACT

An important aspect of attitude determination is covariance analysis. It can be essential for precise and robust instrument pointing, target acquisition and communications. As part of this analysis, covariance visualization can be a powerful tool directly illustrating the spatial structure of the uncertainty. While covariance visualization of position naturally produces ellipsoids in three-dimensional space, covariance visualization of attitude is not as straightforward. Various attitude error parameterizations and corresponding covariance definitions have been introduced for attitude determination in both single-frame and filter configurations. None produce ellipsoids that can be easily interpreted in three-dimensional position space. This paper describes how pointing uncertainty affected by both relative position and attitude can be characterized and visualized using several useful projections, including a unit sphere and a focal plane.

INTRODUCTION

Construction of the attitude covariance matrix depends on the type of attitude representation. A problem arises from the fact that three parameter attitude representations contain singularities and higher-order representations contain constraints.[1] Several methods have been proposed to circumvent this problem,[2-4] one of which is selected in this paper. The selected method assumes that attitude perturbations with respect to its mean estimate can be interpreted as a Gaussian multivariate process using some attitude representation isomorphic to $SO(3)$. In this case, for all practical purposes, only small perturbations with respect to the mean attitude estimate are considered. The Gaussian probability density function (pdf) in the region of small attitude perturbations can be adequately characterized by another Gaussian pdf defined in terms of a three-parameter representation with the singularity far removed from its "zero". Let $\widetilde{\mathbf{C}}$ be the direction cosine matrix that represents stochastic attitude perturbations with respect to its mean estimate. The following first order approximation may be applied for small attitude perturbations:

$$\widetilde{\mathbf{C}} \approx \mathbf{E_3} - \boldsymbol{\delta}\mathbf{C}, \tag{1}$$

where $\mathbf{E_3} \in \Re^{3\times3}$ is the identity matrix,

$$\boldsymbol{\delta}\mathbf{C} \approx \begin{bmatrix} 0 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 0 & -\delta\theta_x \\ -\delta\theta_y & \delta\theta_x & 0 \end{bmatrix} \equiv \boldsymbol{\delta\theta}^{\times}, \tag{2}$$

$$\boldsymbol{\delta\theta} \equiv \begin{bmatrix} \delta\theta_x & \delta\theta_y & \delta\theta_z \end{bmatrix}^{\mathbf{T}} \in \Re^3. \tag{3}$$

The attitude covariance matrix can now be constructed using the three-parameter representation above:

$$\mathbf{P_{\theta\theta}} \equiv E\left\{\boldsymbol{\delta\theta\delta\theta}^{\mathbf{T}}\right\} = \begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{xy} & p_{yy} & p_{yz} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix}, \tag{4}$$

where $E\{\ \}$ denotes the expected value operator. The covariance matrix $\mathbf{P_{\theta\theta}} \in \Re^{3\times3}$ is positive definite and symmetric, $\mathbf{P_{\theta\theta}} = \mathbf{P_{\theta\theta}^T} > \mathbf{0}$, which means that there are only six unique elements in the matrix.

POINTING COVARIANCE

The covariance matrix defined in the previous section can be used directly to define either k-sigma ellipsoids or ellipsoids encompassing certain probability levels. However, since the covariance is not based on position variables, its visualization in three-dimensional position space is not instructive. One of the important practical uses of attitude covariance is for pointing analysis. Two aspects of this analysis will be covered in this paper. First, consider a vector, perturbations of which with respect to its mean estimate follow Gaussian distribution and are uncorrelated with attitude. For example, the mean estimate and covariance of relative position vector between two spacecraft can be obtained from an orbit-determination (OD) process, which has no correlation to the attitude. Then, this stochastic vector and its mean estimate, all originally defined in some known reference frame, can be deterministically mapped into the frame defined by the mean attitude estimate. The vector and its mean are denoted by $\tilde{\mathbf{V}}' \in \Re^3$ and $\overline{\mathbf{V}} \in \Re^3$, respectively. Note that $\Delta\mathbf{V}' \equiv \tilde{\mathbf{V}}' - \overline{\mathbf{V}}$ is the zero mean Gaussian multivariate process. The vector $\tilde{\mathbf{V}}'$ can be in turn mapped into a perturbed attitude frame using the direction cosine matrix $\tilde{\mathbf{C}}$:

$$\begin{aligned}\tilde{\mathbf{V}} &= \tilde{\mathbf{C}}\tilde{\mathbf{V}}' = \tilde{\mathbf{C}}\overline{\mathbf{V}} + \tilde{\mathbf{C}}\Delta\mathbf{V}' \\ &\approx \overline{\mathbf{V}} + \Delta\mathbf{V}' - \delta\mathbf{C}\overline{\mathbf{V}} - \delta\mathbf{C}\Delta\mathbf{V}'\end{aligned} \quad (5)$$

The mean estimate of $\tilde{\mathbf{V}}$ remains the same

$$E\{\tilde{\mathbf{V}}\} = \overline{\mathbf{V}}, \quad (6)$$

while the covariance becomes defined in terms of

$$\Delta\mathbf{V} \equiv \tilde{\mathbf{V}} - \overline{\mathbf{V}} \approx \Delta\mathbf{V}' - \delta\mathbf{C}\overline{\mathbf{V}} - \delta\mathbf{C}\Delta\mathbf{V}'. \quad (7)$$

Using Eq.(2) and truncating at first order yields:

$$\begin{aligned}\Delta\mathbf{V} &\approx \Delta\mathbf{V}' - \delta\boldsymbol{\theta}^{\times}\overline{\mathbf{V}} - \delta\boldsymbol{\theta}^{\times}\Delta\mathbf{V}' \\ &= \Delta\mathbf{V}' + \overline{\mathbf{V}}^{\times}\delta\boldsymbol{\theta} - \delta\boldsymbol{\theta}^{\times}\delta\mathbf{V}' \\ &\approx \Delta\mathbf{V}' + \overline{\mathbf{V}}^{\times}\delta\boldsymbol{\theta}\end{aligned} \quad (8)$$

It is convenient to define the relative stochastic vector

$$\delta\mathbf{V} \equiv \frac{\Delta\mathbf{V}}{\|\overline{\mathbf{V}}\|} \approx \delta\mathbf{V}' + \hat{\overline{\mathbf{V}}}^{\times}\delta\boldsymbol{\theta}, \quad (9)$$

where $\delta\mathbf{V}' \equiv \frac{\Delta\mathbf{V}'}{\|\overline{\mathbf{V}}\|}$ and $\hat{\overline{\mathbf{V}}} \equiv \frac{\overline{\mathbf{V}}}{\|\overline{\mathbf{V}}\|}$. Note that $\delta\mathbf{V}$, $\delta\mathbf{V}'$ and $\delta\boldsymbol{\theta}$ are zero mean Gaussian multivariate processes, and that according to the original assumption $\delta\mathbf{V}'$ and $\delta\boldsymbol{\theta}$ are uncorrelated. Then the following covariance, which takes into account attitude uncertainty, can be found:

$$\mathbf{P}_{\mathbf{VV}} \approx \mathbf{P}'_{\mathbf{VV}} + \hat{\overline{\mathbf{V}}}^{\times}\mathbf{P}_{\boldsymbol{\theta\theta}}\hat{\overline{\mathbf{V}}}^{\times\mathrm{T}}, \quad (10)$$

where

$$\mathbf{P}_{\mathbf{VV}} \equiv E\{\delta\mathbf{V}\delta\mathbf{V}^{\mathrm{T}}\} \in \Re^{3\times3}, \quad (11)$$

$$\mathbf{P}'_{\mathbf{VV}} \equiv E\{\delta\mathbf{V}'\delta\mathbf{V}'^{\mathrm{T}}\} \in \Re^{3\times3}. \quad (12)$$

The resulting covariance $\mathbf{P}_{\mathbf{VV}}$ differs from the original covariance $\mathbf{P}'_{\mathbf{VV}}$, because according to Eq.(10) it is expanded in directions perpendicular to the mean vector estimate. This expansion, due to the added attitude uncertainty, may not simply result in the enlargement of certain dimensions of the corresponding ellipsoids. Depending on the orientation of the original ellipsoid with respect to the mean vector, the resulting ellipsoid may also change in orientation (Fig.1).

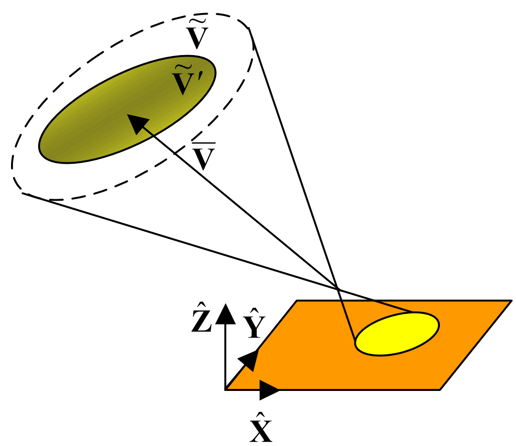

Figure 1 Pointing covariance analysis

The ellipsoids described above in three dimensions are special cases of n-dimensional surfaces of equal probability density, which can be a very useful and visual measure of uncertainty constructed from the covariance matrix. In general, Gaussian pdf for a zero-mean n-dimensional vector $\mathbf{x} \in \Re^n$ is defined as[5]

$$\rho(\mathbf{x}) \equiv \frac{\exp(-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{P}_{\mathbf{xx}}^{-1}\mathbf{x})}{\sqrt{(2\pi)^n|\mathbf{P}_{\mathbf{xx}}|}}, \quad (13)$$

where $\mathbf{P}_{\mathbf{xx}} \equiv E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}$. A surface of equal probability density in n dimensions becomes a hyper-ellipsoid, which encloses an n-dimensional volume and has an associated probability of locating $\mathbf{x}$ inside. The hyper-ellipsoid is defined by the following equation:

$$\mathbf{x}^{\mathbf{T}}\mathbf{P}_{\mathbf{xx}}^{-1}\mathbf{x} = k^2, \tag{14}$$

where k is the scaling factor that determines the size of the hyper-ellipsoid. The hyper-ellipsoid defined by the un-scaled covariance matrix is called 1-sigma. Similarly, k-sigma hyper-ellipsoids can be introduced. Note that k can be derived from the associated probability level of locating $\mathbf{x}$ inside the enclosed volume. The relationship between the scaling factor k and the associated probability $p_n(k)$ also depends on the dimension n:[5]

$$\begin{aligned} p_n(k) &= erf(k/\sqrt{2}) \\ &\quad - \sqrt{2/\pi}\exp(-k^2/2) \\ &\quad \times \sum_{i=1}^{(n-1)/2} \frac{k^{2i-1}}{\prod_{j=1}^{i}(2j-1)} \end{aligned}, \tag{15}$$

for odd $n = 1,3,5,\ldots$ and

$$\begin{aligned} p_n(k) &= 1 - \exp(-k^2/2) \\ &\quad \times \left\{ 1 + \sum_{i=1}^{(n-2)/2} \frac{k^{2i}}{\prod_{j=1}^{i} 2j} \right\} \end{aligned}, \tag{16}$$

for even $n = 2,4,6,\ldots$.

The analysis presented above demonstrates how to quantify the uncertainty in the perceived direction of a stochastic vector as viewed from a stochastic frame. For example, the expected uncertainty in the direction of a relative position vector must be augmented as shown above when it is interpreted for an instrument fixed in the spacecraft body frame, the attitude of which is subject to attitude determination. This discussion leads to the second aspect of the covariance analysis covered in this paper: apart from the ellipsoid itself, its projections onto a unit sphere and the focal plane of an instrument may be of interest. These projections are developed in the next two sections.

COVARIANCE PROJECTION ONTO A UNIT SPHERE

Development of the covariance projection onto a unit sphere can be facilitated by the introduction of a new coordinate frame defined by the right-handed triad of the unit vectors $\{\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\}$ with $\hat{\mathbf{Z}} = \hat{\bar{\mathbf{V}}}$. All of these vectors are defined with respect to the mean attitude frame. By construction, the plane containing vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ is orthogonal to the normalized mean vector $\hat{\bar{\mathbf{V}}}$ and is tangential to the unit sphere at that point. Then, the two- dimensional Cartesian coordinates $\mathbf{d} \in \Re^2$ defined along vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ can be used to characterize projections of the three-dimensional vectors $\tilde{\mathbf{V}}$ and $\bar{\mathbf{V}}$:

$$\tilde{\mathbf{d}} \equiv \begin{bmatrix} \tilde{d}_x \\ \tilde{d}_y \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix} \frac{\tilde{\mathbf{V}}}{\hat{\bar{\mathbf{V}}}^{\mathbf{T}}\tilde{\mathbf{V}}} \tag{17}$$

and

$$\bar{\mathbf{d}} \equiv \begin{bmatrix} \bar{d}_x \\ \bar{d}_y \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix} \hat{\bar{\mathbf{V}}} = \mathbf{0}. \tag{18}$$

Therefore, $\boldsymbol{\delta}\mathbf{d} \equiv \tilde{\mathbf{d}} - \bar{\mathbf{d}} = \tilde{\mathbf{d}}$, and

$$\boldsymbol{\delta}\mathbf{d} = \begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix} \frac{\hat{\bar{\mathbf{V}}} + \boldsymbol{\delta}\mathbf{V}}{1 + \hat{\bar{\mathbf{V}}}^{\mathbf{T}}\boldsymbol{\delta}\mathbf{V}} \approx \begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix} \boldsymbol{\delta}\mathbf{V}. \tag{19}$$

Equivalently, in compact form,

$$\boldsymbol{\delta}\mathbf{d} = \mathbf{H}\delta\mathbf{V}, \tag{20}$$

where $\mathbf{H} \in \Re^{2\times 3}$ is the projection matrix defined as

$$\mathbf{H} \equiv \mathbf{H}(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}) \equiv \begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix}. \tag{21}$$

Note that, according to Eq.(20), $\boldsymbol{\delta}\mathbf{d}$ is also a zero mean Gaussian multivariate process:

$$E\{\boldsymbol{\delta}\mathbf{d}\} = \mathbf{H}E\{\boldsymbol{\delta}\mathbf{V}\} = \mathbf{0}. \tag{22}$$

Also, note that

$$\mathbf{H}\mathbf{H}^{\mathbf{T}} = \mathbf{E}_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{23}$$

so that the pseudo-inverse $\mathbf{H}^{+} \in \Re^{3\times 2}$ is greatly simplified:

$$\mathbf{H}^{+} \equiv \mathbf{H}^{\mathbf{T}}\left(\mathbf{H}\mathbf{H}^{\mathbf{T}}\right)^{-1} = \mathbf{H}^{\mathbf{T}} = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}}\end{bmatrix}. \quad (24)$$

It is instructive to use Eq.(9) to identify contributions to the projection from the position covariance and from the attitude covariance:

$$\begin{aligned}\mathbf{P}_{\mathbf{dd}} &\equiv E\left\{\boldsymbol{\delta}\mathbf{d}\boldsymbol{\delta}\mathbf{d}^{\mathbf{T}}\right\} = \mathbf{H}\mathbf{P}_{\mathbf{VV}}\mathbf{H}^{\mathbf{T}} \\ &= \mathbf{H}\mathbf{P}'_{\mathbf{VV}}\mathbf{H}^{\mathbf{T}} + \mathbf{G}\mathbf{P}_{\boldsymbol{\theta\theta}}\mathbf{G}^{\mathbf{T}}\end{aligned} , \quad (25)$$

where $\mathbf{G} \in \Re^{2\times 3}$ is defined as

$$\begin{aligned}\mathbf{G} &\equiv \mathbf{G}(\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}}) = -\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\hat{\overline{\mathbf{V}}}^{\times} \\ &= \begin{bmatrix}\left(\hat{\overline{\mathbf{V}}} \times \hat{\mathbf{X}}\right)^{\mathbf{T}} \\ \left(\hat{\overline{\mathbf{V}}} \times \hat{\mathbf{Y}}\right)^{\mathbf{T}}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{Y}}^{\mathbf{T}} \\ -\hat{\mathbf{X}}^{\mathbf{T}}\end{bmatrix}\end{aligned} \quad (26)$$

and $\mathbf{G}\mathbf{G}^{\mathbf{T}} = \mathbf{E}_{\mathbf{2}}$, $\mathbf{G}^{+} = \mathbf{G}^{\mathbf{T}} = \begin{bmatrix}\hat{\mathbf{Y}} & -\hat{\mathbf{X}}\end{bmatrix} \in \Re^{3\times 2}$.

COVARIANCE PROJECTION ONTO A FOCAL PLANE

A focal plane projection is similar to the projection onto a plane tangential to the unit sphere developed in the previous section. However, unlike the tangential plane, the focal plane defined by the right-handed triad $\left\{\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}}\right\}$ is perpendicular to the unit vector $\hat{\mathbf{Z}}$ and is not necessarily perpendicular to the vector $\overline{\mathbf{V}}$. In addition, the focal ratio f scales the coordinates in the focal plane. In this case, the two-dimensional Cartesian coordinates $\mathbf{d} \in \Re^{2}$ are derived from the three dimensional vectors $\widetilde{\mathbf{V}}$ and $\overline{\mathbf{V}}$ as follows:

$$\widetilde{\mathbf{d}} \equiv \mathbf{d}_{\mathbf{0}} - f\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\frac{\widetilde{\mathbf{V}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\widetilde{\mathbf{V}}}, \quad (27)$$

where $\hat{\mathbf{Z}}^{\mathbf{T}}\widetilde{\mathbf{V}} > 0$, and

$$\overline{\mathbf{d}} \equiv \mathbf{d}_{\mathbf{0}} - f\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\frac{\overline{\mathbf{V}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\overline{\mathbf{V}}}, \quad (28)$$

where $\hat{\mathbf{Z}}^{\mathbf{T}}\overline{\mathbf{V}} > 0$. In this case,

$$\begin{aligned}\boldsymbol{\delta}\mathbf{d} &\equiv \widetilde{\mathbf{d}} - \overline{\mathbf{d}} \\ &= f\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\left(\frac{\overline{\mathbf{V}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\overline{\mathbf{V}}} - \frac{\widetilde{\mathbf{V}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\widetilde{\mathbf{V}}}\right) \\ &= f\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\frac{\overline{\mathbf{V}}\hat{\mathbf{Z}}^{\mathbf{T}}\widetilde{\mathbf{V}} - \widetilde{\mathbf{V}}\hat{\mathbf{Z}}^{\mathbf{T}}\overline{\mathbf{V}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\overline{\mathbf{V}}\hat{\mathbf{Z}}^{\mathbf{T}}\widetilde{\mathbf{V}}} \\ &\approx \begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\frac{f}{\hat{\mathbf{Z}}^{\mathbf{T}}\hat{\overline{\mathbf{V}}}}\left(\frac{\hat{\overline{\mathbf{V}}}\hat{\mathbf{Z}}^{\mathbf{T}}}{\hat{\mathbf{Z}}^{\mathbf{T}}\hat{\overline{\mathbf{V}}}} - \mathbf{E}_{\mathbf{3}}\right)\boldsymbol{\delta}\mathbf{V}\end{aligned} , \quad (29)$$

Equivalently, in compact form,

$$\boldsymbol{\delta}\mathbf{d} = \mathbf{H}\delta\mathbf{V}, \quad (30)$$

where, in this case, the projection matrix $\mathbf{H} \in \Re^{2\times 3}$ is defined as

$$\begin{aligned}\mathbf{H} &\equiv \mathbf{H}(\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}},f,\hat{\overline{\mathbf{V}}}) \\ &\equiv \frac{-f}{\hat{\overline{V}}_z}\begin{bmatrix}\hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}}\end{bmatrix}\left(\mathbf{E}_{\mathbf{3}} - \frac{\hat{\overline{\mathbf{V}}}\hat{\mathbf{Z}}^{\mathbf{T}}}{\hat{\overline{V}}_z}\right) \\ &= \frac{-f}{\hat{\overline{V}}_z^{2}}\begin{bmatrix}\hat{\overline{V}}_z\hat{\mathbf{X}}^{\mathbf{T}} - \hat{\overline{V}}_x\hat{\mathbf{Z}}^{\mathbf{T}} \\ \hat{\overline{V}}_z\hat{\mathbf{Y}}^{\mathbf{T}} - \hat{\overline{V}}_y\hat{\mathbf{Z}}^{\mathbf{T}}\end{bmatrix}\end{aligned} \quad (31)$$

with $\hat{\overline{V}}_x \equiv \hat{\mathbf{X}}^{\mathbf{T}}\hat{\overline{\mathbf{V}}}$, $\hat{\overline{V}}_y \equiv \hat{\mathbf{Y}}^{\mathbf{T}}\hat{\overline{\mathbf{V}}}$, $\hat{\overline{V}}_z \equiv \hat{\mathbf{Z}}^{\mathbf{T}}\hat{\overline{\mathbf{V}}}$.

As in the previous section, $\boldsymbol{\delta}\mathbf{d}$ is a zero mean Gaussian multivariate process:

$$E\left\{\boldsymbol{\delta}\mathbf{d}\right\} = \mathbf{H}E\left\{\boldsymbol{\delta}\mathbf{V}\right\} = \mathbf{0}. \quad (32)$$

The pseudo-inverse $\mathbf{H}^{+} \in \Re^{3\times 2}$ of the projection matrix is still relatively simple:

$$\begin{aligned}\mathbf{H}^{+} &\equiv \mathbf{H}^{\mathbf{T}}\left(\mathbf{H}\mathbf{H}^{\mathbf{T}}\right)^{-1} = -(1/f) \\ &\times\begin{bmatrix}\hat{\overline{V}}_z\hat{\mathbf{X}} - \hat{\overline{V}}_x\hat{\mathbf{Z}} & \hat{\overline{V}}_z\hat{\mathbf{Y}} - \hat{\overline{V}}_y\hat{\mathbf{Z}}\end{bmatrix} \\ &\times\begin{bmatrix}1 - \hat{\overline{V}}_x^{2} & -\hat{\overline{V}}_x\hat{\overline{V}}_y \\ -\hat{\overline{V}}_x\hat{\overline{V}}_y & 1 - \hat{\overline{V}}_y^{2}\end{bmatrix}\end{aligned} , \quad (33)$$

which is based on

$$\mathbf{H}\mathbf{H}^{\mathbf{T}} = \left(\frac{f}{\hat{\bar{V}}_z}\right)^2 \mathbf{E_2}$$

$$+\left(\frac{f}{\hat{\bar{V}}_z}\right)^2 \frac{1}{\hat{\bar{V}}_z^2}\begin{bmatrix} \hat{\bar{V}}_x^2 & \hat{\bar{V}}_x\hat{\bar{V}}_y \\ \hat{\bar{V}}_x\hat{\bar{V}}_y & \hat{\bar{V}}_y^2 \end{bmatrix}, \tag{34}$$

$$\left(\mathbf{H}\mathbf{H}^{\mathbf{T}}\right)^{-1} = \left(\hat{\bar{V}}_z / f\right)^2$$

$$\times \begin{bmatrix} 1-\hat{\bar{V}}_x^2 & -\hat{\bar{V}}_x\hat{\bar{V}}_y \\ -\hat{\bar{V}}_x\hat{\bar{V}}_y & 1-\hat{\bar{V}}_y^2 \end{bmatrix}. \tag{35}$$

Similar to the previous section, it is instructive to develop formulation for the projected covariance that identifies contributions from the position covariance and from the attitude covariance:

$$\begin{aligned}\mathbf{P_{dd}} &\equiv E\left\{\boldsymbol{\delta}\mathbf{d}\boldsymbol{\delta}\mathbf{d}^{\mathbf{T}}\right\} = \mathbf{H}\mathbf{P_{VV}}\mathbf{H}^{\mathbf{T}} \\ &= \mathbf{H}\mathbf{P}'_{\mathbf{VV}}\mathbf{H}^{\mathbf{T}} + \mathbf{G}\mathbf{P}_{\boldsymbol{\theta\theta}}\mathbf{G}^{\mathbf{T}}\end{aligned}, \tag{36}$$

where $\mathbf{G} \in \Re^{2\times 3}$ is defined as

$$\begin{aligned}\mathbf{G} &\equiv \mathbf{G}(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}, f, \hat{\bar{\mathbf{V}}}) \\ &\equiv \frac{f}{\hat{\bar{V}}_z}\begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix}\left(\mathbf{I_3} - \frac{\hat{\bar{\mathbf{V}}}\hat{\mathbf{Z}}^{\mathbf{T}}}{\hat{\bar{V}}_z}\right)\hat{\bar{\mathbf{V}}}^{\times} \\ &= \frac{f}{\hat{\bar{V}}_z}\begin{bmatrix} \left(\hat{\mathbf{X}}\times\hat{\bar{\mathbf{V}}}\right)^{\mathbf{T}} - \frac{\hat{\bar{V}}_x}{\hat{\bar{V}}_z}\left(\hat{\mathbf{Z}}\times\hat{\bar{\mathbf{V}}}\right)^{\mathbf{T}} \\ \left(\hat{\mathbf{Y}}\times\hat{\bar{\mathbf{V}}}\right)^{\mathbf{T}} - \frac{\hat{\bar{V}}_y}{\hat{\bar{V}}_z}\left(\hat{\mathbf{Z}}\times\hat{\bar{\mathbf{V}}}\right)^{\mathbf{T}} \end{bmatrix}\end{aligned}. \tag{37}$$

Note that the expressions above reduce to forms very similar to those of the planar projections in the previous section if the focal plane is perpendicular to the mean vector estimate, i.e. $\hat{\mathbf{Z}}$ is collinear with $\bar{\mathbf{V}}$:

$$\mathbf{H}_{\perp} \equiv \mathbf{H}(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}, f, \hat{\mathbf{Z}}) = -f\begin{bmatrix} \hat{\mathbf{X}}^{\mathbf{T}} \\ \hat{\mathbf{Y}}^{\mathbf{T}} \end{bmatrix}, \tag{38}$$

$$\mathbf{H}_{\perp}\mathbf{H}_{\perp}^{\mathbf{T}} = f^2\mathbf{E_2} \tag{39}$$

and

$$\mathbf{G}_{\perp} \equiv \mathbf{G}(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}, f, \hat{\mathbf{Z}}) = -f\begin{bmatrix} \hat{\mathbf{Y}}^{\mathbf{T}} \\ -\hat{\mathbf{X}}^{\mathbf{T}} \end{bmatrix}, \tag{40}$$

$$\mathbf{G}_{\perp}\mathbf{G}_{\perp}^{\mathbf{T}} = f^2\mathbf{E_2}. \tag{41}$$

Finally, note that the tangential plane projection discussed in the previous section can be viewed as the special case of the focal plane projection with $\hat{\mathbf{Z}} = \hat{\bar{\mathbf{V}}}$, $\mathbf{d_0} = \mathbf{0}$ and $f = -1$.

SURFACES OF EQUAL PROBABILITY DENSITY

The projections of the covariance matrix established in the previous sections are necessary but not sufficient for creating k-sigma ellipses in two dimensions. These ellipses (similar to ellipsoids in three or more dimensions (Eq.(14)) are based on the following equation:

$$\boldsymbol{\delta}\mathbf{d}^{\mathbf{T}}\mathbf{P}_{\mathbf{dd}}^{-1}\boldsymbol{\delta}\mathbf{d} = k^2, \tag{42}$$

where $\mathbf{P}_{\mathbf{dd}}^{-1} = \mathbf{P}_{\mathbf{dd}}^{-\mathbf{T}} > 0, \mathbf{P}_{\mathbf{dd}}^{-1} \in \Re^{2\times 2}$ and k is the specified scaling factor, which can be related to the associated probability as described above in this paper (Eqs.(15,16)). The equation for ellipses requires computation of the inverse of the covariance matrix in two dimensions. The first and most straightforward approach is to simply invert the projected covariance matrix:

$$\mathbf{P}_{\mathbf{dd}}^{-1} = \left[\mathbf{P}_{\mathbf{dd}}^{-1}\right]_{\mathbf{1}} \equiv \left[\mathbf{H}\mathbf{P_{VV}}\mathbf{H}^{\mathbf{T}}\right]^{-1}. \tag{43}$$

Since the covariance matrix is projected first, the resulting inverse does not account for any correlation with the discarded dimension. This also means that the relationship between the scaling factor k and the associated probability, which is based on Eq.(16), must be established in two dimensions: $p_2(k)$.[5] Alternatively, the covariance matrix can be first inverted and then projected:

$$\mathbf{P}_{\mathbf{dd}}^{-1} \equiv \left[\mathbf{P}_{\mathbf{dd}}^{-1}\right]_{\mathbf{2}} \equiv \left[\mathbf{H}^{+}\right]^{\mathbf{T}}\mathbf{P}_{\mathbf{VV}}^{-1}\mathbf{H}^{+}. \tag{44}$$

This second method does take into account correlations with the discarded dimension. Thus, the relationship between the scaling factor k and the associated probability, which in this case is based on Eq.(15), must be established in three dimensions: $p_3(k)$.[5] Note that, in general for non-diagonal matrices, the two methods do not generate the same inverses: $\left[\mathbf{P}_{\mathbf{dd}}^{-1}\right]_{\mathbf{1}} \neq \left[\mathbf{P}_{\mathbf{dd}}^{-1}\right]_{\mathbf{2}}$. Geometrically, the first method uses the extent of an entire ellipsoid to create the corresponding projected ellipse. The second method is based on the cross-section of the ellipsoid defined by the plane crossing through its center.

VISUALIZATION

For visualization, the size and orientation of the two dimensional ellipse resulting from either of the two methods can be determined using eigen-decomposition[6] of the inverse $\mathbf{P}_{\mathbf{dd}}^{-1}$:

$$\mathbf{P}_{\mathbf{dd}}^{-1} = \mathbf{Q}\mathbf{\Sigma}^{-2}\mathbf{Q}^{\mathrm{T}} , \qquad (45)$$

where $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}, \mathbf{Q} \equiv \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \in \Re^{2\times 2}$ is the orthogonal matrix of eigenvectors and $\mathbf{\Sigma}^{-2} \equiv \begin{bmatrix} \varepsilon_1^{-2} & 0 \\ 0 & \varepsilon_2^{-2} \end{bmatrix} \in \Re^{2\times 2}$ is the diagonal matrix of eigenvalues. Then, the ellipse defined in the $\hat{\mathbf{X}}$-$\hat{\mathbf{Y}}$ plane is centered at the projection of the mean estimate $\overline{\mathbf{d}}$ and has semi-axes of the following sizes:

$$\sigma_i = k\varepsilon_i, i = 1,2 . \qquad (46)$$

The orientation of the ellipse can be determined based on the angle $0 \le \alpha < 2\pi$ measured from the unit vector $\hat{\mathbf{X}}$ to the first semi-axis of the ellipse, i.e. the semi-axis, size of which is equal to σ_1 :

$$\alpha = \tan^{-1}\{q_{12} / q_{22}\}. \qquad (47)$$

The contour on the unit sphere that follows the ellipse on the tangential plane can be generated using the following two angular variables:

$$\delta\varphi = \tan^{-1}(\delta d_y / \delta d_x), \qquad (48)$$

$$\delta\lambda = \tan^{-1}\|\boldsymbol{\delta}\mathbf{d}\|, \qquad (49)$$

where $0 \le \delta\varphi < 2\pi$ is the clock-angle measured counterclockwise about the vector $\overline{\mathbf{V}}$ from the unit vector $\hat{\mathbf{X}}$ and the small perturbation angle $0 < \delta\lambda << \pi$ is measured away from the vector $\overline{\mathbf{V}}$.

Several issues related to interpolation of the resulting matrices have been discussed recently for position covariance visualization in three dimensions.[7] Similar issues arise in two dimensions and similar solutions are proposed in this paper. Since there is no guarantee that the first eigenvalue obtained during eigen-decomposition at one time will correspond to the first eigenvalue obtained at a later time, the eigenvalues are consistently ordered according to their size. Symmetry of the ellipse also produces a 180° ambiguity in its orientation. To avoid possible 180° rotations when interpolating subsequent ellipses, the orientation is selected such that the angle between them is minimized.

CONCLUSIONS

A method for combining the position and attitude covariance matrices to characterize pointing uncertainty has been presented. Surfaces of equal probability density have been generated in both two and three dimensions based on the pointing covariance. The two dimensional surfaces have been generated using projections onto a unit sphere and a focal plane of an instrument. These quantities, which are well suited for visualization, provide additional insight into a geometric structure of the pointing uncertainty and its evolution with time.

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