GENERALIZATION OF ADAPTIVE ATTITUDE TRACKING

S. Tanygin
Member AIAA
Lead Engineer, Attitude Dynamics and Control
Analytical Graphics, Inc.
Malvern, PA

ABSTRACT

The paper introduces several generalizations of the attitude tracking problem for a rigid-body. The existing control laws are reformulated to include momentum bias both with and without angular velocity measurements. Then standard model reference adaptive control (MRAC) techniques are employed to provide adaptation in the presence of the unknown constant inertia matrix. The analysis is performed using Lyapunov stability methods and their extensions. The control laws are evaluated independently from a specific attitude representation, which facilitates development of control laws for any given attitude representation. The quaternion based control law is provided as an example and evaluated using numerical simulation for a realistic mission profile.

INTRODUCTION

The attitude dynamics and control of a rigid body have been studied by many authors from both theoretical and practical standpoint.[1-10] The attitude kinematics and rigid-body dynamics represent one of the classical examples of cascade passive nonlinear systems linear in control, which can be stabilized by very simple linear or almost linear feedback control laws.[1-10] What is more, a passive nature of both dynamics and kinematics permits the angular velocity measurements to be replaced by the outputs of a stable linear system driven by the chosen attitude representation.[3,5,7-10] The results developed for the attitude stabilization are easily extended to tracking using relative formulations for the kinematics and dynamics.[7-10] However, note that such control laws, particularly recently proposed angular velocity free control laws, [7,10] are inherently continuous functions of the tracking errors. This presents a practical problem, because using the rigid-body dynamics prohibits application of momentum exchange devices, which are well suited for producing continuous control. When angular velocity measurements are available, it is very straightforward to include the momentum bias into both the dynamics and the control law. When angular velocity measurements are not available, maintaining angular velocity free formulation of the control law appears not as straightforward. Another common feature of attitude tracking control laws is their dependency of the inertia matrix. This makes them susceptible to errors in the estimated inertia matrix provided for control design. A well-known alternative is to consider adaptive control, specifically to use MRAC techniques and to include additional variables that can be adapted to guarantee overall stability even in the presence of errors in the estimated inertia matrix.[11] Lyapunov methods and their extensions can be used to evaluate stability of both adaptive and non-adaptive control laws. Note that passivity based control laws rely on LaSalle's invariance principle to determine stability based on the largest invariant set of the closed loop system. Introduction of additional variables for adaptive control complicates the analysis and may change the largest invariant set attainable by the attitude error. The analysis does not need to be performed using a specific attitude representation. Instead, if it is done generically assuming some form of attitude representation isomorphic to $SO(3)$,[12,13] the results can provide a greater insight into the nature of attitude motion and the actual control law can be formulated once the attitude representation is selected.

Nomenclature

A cross-product of two three-dimensional vectors $\mathbf{a}\times\mathbf{b}$ can be represented as the matrix-vector product $\mathbf{a}^{\times}\mathbf{b}$, where $\mathbf{a}^{\times} \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \Re^{3\times 3}$ is the skew-symmetric matrix constructed from the elements of vector $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$, where $\mathbf{a},\mathbf{b} \in \Re^3$ and both are expressed in the frame $\{\hat{\mathbf{e}}_\mathbf{i}\}$, which denotes a triad of mutually orthogonal unit vectors.

A matrix-vector product of a symmetric matrix $\mathbf{K} = \mathbf{K}^\mathbf{T} \in \Re^{3\times 3}$ and a vector $\mathbf{r} \in \Re^3$ can be represented as a different matrix vector product:

$$\mathbf{Kr} \equiv \mathbf{r}^{\otimes}\mathbf{K}^{\oplus}, \tag{1}$$

where

$$\mathbf{K}^{\oplus} \equiv \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{22} & K_{23} & K_{33} \end{bmatrix}^{\mathrm{T}}, \quad (2)$$

$$\mathbf{K} \equiv \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix}, \quad (3)$$

$$\mathbf{r}^{\otimes} \equiv \begin{bmatrix} r_1 & r_2 & r_3 & 0 & 0 & 0 \\ 0 & r_1 & 0 & r_2 & r_3 & 0 \\ 0 & 0 & r_1 & 0 & r_2 & r_3 \end{bmatrix}, \quad (4)$$

$$\mathbf{r} \equiv \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^{\mathrm{T}}. \quad (5)$$

Three types of frames are used in this paper. The *body frame* $\{\hat{\mathbf{b}}_{\mathbf{i}}\}$ is attached to a rotating rigid body. The *reference* or *desired frame* $\{\hat{\mathbf{r}}_{\mathbf{i}}\}$ is a frame with respect to which attitude errors are found. The *inertial frame* $\{\hat{\mathbf{i}}_{\mathbf{i}}\}$ provides a common inertial reference for all other frames. Unless stated otherwise, it is assumed that all variables referring to the motion of the reference frame with respect to the inertial frame are denoted with a "bar" (e.g. $\overline{\boldsymbol{\omega}}$), all variables referring to the body motion with respect to the inertial frame use capital letters (e.g. $\boldsymbol{\Omega}$) and all variables referring to the motion of the body frame with respect to the reference frame are also called attitude *error variables* and do not have a special designation (e.g. $\boldsymbol{\omega}$). The variables commonly found in any of these categories are some attitude representation $\mathbf{s}$ belonging to a group isomorphic to $SO(3)$ and angular velocity $\boldsymbol{\omega} \in \Re^3$, which is always expressed in the frame, motion of which it describes. Note that differentiation of the angular velocity with respect to time $\dot{\boldsymbol{\omega}} \in \Re^3$ is also carried out in this frame. Additional notation includes transformation to a rotation matrix $\mathbf{C}(\mathbf{s}) \in SO(3)$, the exact form of which depends on the attitude representation $\mathbf{s}$. Also, depending on this representation are the identity attitude $\mathbf{1}_{\mathbf{s}}$ and the inverse attitude $\mathbf{s}^{-1}$, such that $\mathbf{s}^{-1} \circ \mathbf{s} = \mathbf{s} \circ \mathbf{s}^{-1} = \mathbf{1}_{\mathbf{s}}$, where $\circ$ denotes this group's composition operation. The symmetric positive definite and constant inertia matrix $\mathbf{I} \in \Re^{3\times3} : \mathbf{I} = \mathbf{I}^{\mathrm{T}} > \mathbf{0}$ is defined in the body frame. Attitude trajectories and torque are assumed to be functions of time $t \geq t_0$ unless stated otherwise. The relationship between attitude trajectories in different frames is given below:

$$\mathbf{S} = \mathbf{s} \circ \overline{\mathbf{s}}, \quad (6)$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \boldsymbol{\eta}, \quad (7)$$

where $\boldsymbol{\eta} \equiv \mathbf{C}(\mathbf{s})\overline{\boldsymbol{\omega}}$.

Additional nomenclature is introduced as needed throughout this paper.

ADAPTIVE TRACKING WITH MOMENTUM BIAS AND ANGULAR VELOCITY MEASUREMENTS

This section describes attitude dynamics of a rigid-body with momentum bias and introduces control laws that provide desired error dynamics and in the presence of unknown inertia matrix. The control laws are not formulated explicitly and neither are attitude representations and the associated kinematics; instead, the stability of the entire class of control laws is evaluated using Lyapunov methods.

Attitude motion with respect to inertial frame $\{\hat{\mathbf{i}}_{\mathbf{i}}\}$ is governed by the following differential equations:

$$\dot{\mathbf{S}} = \mathbf{p}(\mathbf{S})\boldsymbol{\Omega}, \quad (8)$$

$$\mathbf{I}\dot{\boldsymbol{\Omega}} = -\boldsymbol{\Omega}^{\times}\left[\mathbf{I}\boldsymbol{\Omega} + \mathbf{h}\right] + \mathbf{f}(\mathbf{S},\boldsymbol{\Omega}) + \mathbf{d}(\mathbf{S},\boldsymbol{\Omega}), \quad (9)$$

where $\mathbf{f}(\mathbf{S},\boldsymbol{\Omega}) \equiv \mathbf{g}(\mathbf{S},\boldsymbol{\Omega}) - \dot{\mathbf{h}}(\mathbf{S},\boldsymbol{\Omega})$, the exact form of kinematics $\mathbf{p}(\mathbf{S})$ depends on the particular attitude representation, $\mathbf{S}$; $\mathbf{g}(\mathbf{S},\boldsymbol{\Omega}), \mathbf{d}(\mathbf{S},\boldsymbol{\Omega}) \in \Re^3$ are the net control torque and net disturbance torque applied to the body, both expressed in the body frame; finally, $\mathbf{h} \in \Re^3$ is the momentum bias also expressed in the body frame. Using relationships (Eqs.(6,7)) between the inertial frame $\{\hat{\mathbf{i}}_{\mathbf{i}}\}$ and the reference frame $\{\hat{\mathbf{r}}_{\mathbf{i}}\}$, differential equations governing attitude motion can be re-written in terms of attitude errors with respect to the reference frame:[7,9]

$$\dot{\mathbf{s}} = \mathbf{p}(\mathbf{s})\boldsymbol{\omega}, \quad (10)$$

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\Omega}^{\times}\left[\mathbf{I}\boldsymbol{\Omega} + \mathbf{h}\right] - \overline{\mathbf{H}} + \overline{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega}) + \mathbf{d}(\mathbf{S},\boldsymbol{\Omega}), \quad (11)$$

where $\overline{\mathbf{H}} \equiv \mathbf{I}\left[\mathbf{C}(\mathbf{s})\dot{\overline{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\boldsymbol{\Omega}\right]$ is the correction due to angular motion and acceleration of the reference frame and $\overline{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ is a different formulation of the control law, which should provide stability of the error trajectory $(\mathbf{s},\boldsymbol{\omega})$. This new formulation $\overline{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ can be obtained directly from the original control law $\mathbf{f}(\mathbf{S},\mathbf{\Omega})$ as shown next.

Assume that stability of the trajectory $(\mathbf{S},\mathbf{\Omega})$ governed by Eqs.(8,9) with the control law $\mathbf{f}(\mathbf{S},\mathbf{\Omega})$ can be shown using Lyapunov method. In other words, assume that there exist a proper positive definite Lyapunov function $V_0(\mathbf{S},\mathbf{\Omega}) > 0$ such that has a negative semi-definite derivative $W_0(\mathbf{S},\mathbf{\Omega}) \equiv \dot{V}_0(\mathbf{S},\mathbf{\Omega}) \le 0$ along the trajectories governed by Eqs.(8,9). Note that an entire class of passivity based control laws relies on LaSalle's invariance principle to demonstrate stability in the presence of not strictly negative definite derivative $W_0(\mathbf{S},\mathbf{\Omega})$.

Generalization of the control law $\mathbf{f}(\mathbf{S},\mathbf{\Omega})$ to attitude tracking with respect to non-inertial reference frame $\{\hat{\mathbf{r}}_\mathbf{i}\}$ is straightforward.[7,9] The control law needs to be evaluated along the error trajectory $(\mathbf{s},\boldsymbol{\omega})$ and needs to be modified using the following mapping:

$$\begin{aligned}\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega}) &\equiv \mathbf{f}(\mathbf{s},\boldsymbol{\omega}) \\ &+ \mathbf{\Omega}^{\times}\left[\mathbf{I\Omega}+\mathbf{h}\right] \\ &- \boldsymbol{\omega}^{\times}\left[\mathbf{I}\boldsymbol{\omega}+\mathbf{h}\right] + \overline{\mathbf{H}}\end{aligned} . \tag{12}$$

Note that this mapping is linear with respect to the control law $\mathbf{f}(\mathbf{s},\boldsymbol{\omega})$ it acts on, but is nonlinear with respect to the reference trajectory in terms of $\dot{\bar{\boldsymbol{\omega}}}$ and $\bar{\boldsymbol{\omega}}$. Also, note that mapping needed to represent attitude trajectory with respect to another inertial frame becomes the identity mapping:

$$\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega}) = \mathbf{f}(\mathbf{s},\boldsymbol{\omega}) ,\ \dot{\bar{\boldsymbol{\omega}}} = \bar{\boldsymbol{\omega}} = \mathbf{0} . \tag{13}$$

Stability of the error trajectory $(\mathbf{s},\boldsymbol{\omega})$ governed by Eqs.(10,11) with control law $\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ can be shown using the same Lyapunov function formulation $V_0(\mathbf{s},\boldsymbol{\omega})$. This is because using $\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ in Eqs.(10,11) reduces them to the same form as that of Eqs.(8,9) with the exception of bounded disturbance $\mathbf{d}(\mathbf{S},\mathbf{\Omega})$. In other words, with the exception of bounded disturbance, using $\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ in Eqs.(10,11) allows them to be converted to Eqs.(8,9) by a simple substitution of $(\mathbf{S},\mathbf{\Omega})$ in place of $(\mathbf{s},\boldsymbol{\omega})$.

The following sections in this paper take advantage of a more specific form of the Lyapunov function that simplifies design of passivity based control laws:

$$V_0(\mathbf{s},\boldsymbol{\omega}) \equiv \frac{1}{2}\boldsymbol{\omega}^{\mathbf{T}}\mathbf{I}\boldsymbol{\omega} + V_s(\mathbf{s}) . \tag{14}$$

In the absence of disturbances, derivative of this function evaluated along the trajectories governed by Eqs.(10,11) with the control law $\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$ becomes:

$$\begin{aligned}W_0(\mathbf{s},\boldsymbol{\omega}) &\equiv \boldsymbol{\omega}^{\mathbf{T}}\mathbf{f}(\mathbf{s},\boldsymbol{\omega}) + \dot{\mathbf{s}}^{\mathbf{T}}\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) \\ &= \boldsymbol{\omega}^{\mathbf{T}}\left[\mathbf{f}(\mathbf{s},\boldsymbol{\omega}) + \mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s})\right]\end{aligned} , \tag{15}$$

where $\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) \equiv \left[\dfrac{\partial V_s(\mathbf{s})}{\partial \mathbf{s}}\right]^{\mathbf{T}}$.

Hence, the following generic form of the passivity based control law can be recommended:

$$\mathbf{f}(\mathbf{s},\boldsymbol{\omega}) \equiv -\mathbf{K}\boldsymbol{\omega} - \mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) , \tag{16}$$

where $\mathbf{K} > 0$ and $\mathbf{p}(\mathbf{s}), \boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s})$ are such that $\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) = \mathbf{0}$ implies $\mathbf{s} = \mathbf{1}_\mathbf{s}$. Application of this control law leads to the negative semi-definite derivative

$$W_0(\mathbf{s},\boldsymbol{\omega}) \equiv -\boldsymbol{\omega}^{\mathbf{T}}\mathbf{K}\boldsymbol{\omega} \le 0 . \tag{17}$$

Note that all solutions are bounded, $\mathbf{s},\boldsymbol{\omega} \in \ell_\infty$, because $V_0(\mathbf{s},\boldsymbol{\omega})$ is radially unbounded and $W_0(\mathbf{s},\boldsymbol{\omega}) \le 0$. The set $\mathrm{N} = \{(\mathbf{s},\boldsymbol{\omega}) : W_0 = 0\}$ contains trajectories with $\boldsymbol{\omega} = \mathbf{0}$, which leads to $\dot{\mathbf{s}} = \mathbf{0}$ based on Eq.(10) assuming $\mathbf{p}(\mathbf{s}) \in \ell_\infty$. Trajectories in this set must also maintain zero higher derivatives, e.g. $\dot{\boldsymbol{\omega}} = \mathbf{0}$, hence, based on Eqs.(11,12,16), in the absence of disturbances:

$$\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) = \mathbf{0} , \tag{18}$$

which means $\mathbf{s} = \mathbf{1}_\mathbf{s}$. Therefore, the largest invariant set in N is the set $\mathrm{M} = \{(\mathbf{s},\boldsymbol{\omega}) : \mathbf{s} = \mathbf{1}_\mathbf{s}, \boldsymbol{\omega} = \mathbf{0}\}$. According to LaSalle's invariance principle the system governed by Eqs.(10,11,12,16) is globally asymptotically stable. In particular, all trajectories of the system asymptotically approach M, i.e.

$$\lim_{t\to\infty}\mathbf{s}(\mathbf{t}) = \mathbf{1}_\mathbf{s}, \lim_{t\to\infty}\boldsymbol{\omega}(\mathbf{t}) = \mathbf{0} .$$

This stability result can be extended to adaptive control. Straightforward application of MRAC principles calls for the augmentation of the original Lyapunov function $V_0(\mathbf{s},\boldsymbol{\omega})$ with the following terms:

$$V_I(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \equiv \frac{1}{2}\Delta\mathbf{I}^{\oplus\mathbf{T}}\boldsymbol{\Gamma}\Delta\mathbf{I}^{\oplus}, \qquad (19)$$

where $\boldsymbol{\Gamma} > 0$,

$$\Delta\mathbf{I}^{\oplus} \equiv \hat{\mathbf{I}}^{\oplus} - \mathbf{I}^{\oplus}, \qquad (20)$$

and for the application of the control law $\bar{\mathbf{f}}_{\mathbf{t}}(\mathbf{s},\boldsymbol{\omega},\hat{\mathbf{I}})$ similar to $\bar{\mathbf{f}}(\mathbf{s},\boldsymbol{\omega})$, but in which true inertia matrix $\mathbf{I}$ is replaced with its estimate $\hat{\mathbf{I}}$:

$$\begin{aligned}\bar{\mathbf{f}}_{\mathbf{t}}(\mathbf{s},\boldsymbol{\omega},\hat{\mathbf{I}}) &\equiv \mathbf{f}(\mathbf{s},\boldsymbol{\omega}) \\ &+ \boldsymbol{\Omega}^{\times}\left[\hat{\mathbf{I}}\boldsymbol{\Omega} + \mathbf{h}\right] \\ &- \boldsymbol{\omega}^{\times}\left[\hat{\mathbf{I}}\boldsymbol{\omega} + \mathbf{h}\right] + \hat{\overline{\mathbf{H}}}\end{aligned}, \qquad (21)$$

with $\hat{\overline{\mathbf{H}}} \equiv \hat{\mathbf{I}}\left[\mathbf{C}(\mathbf{s})\dot{\overline{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\boldsymbol{\Omega}\right]$. The adaptive law is then selected so as to cancel all inertia matrix dependent terms from the derivative of the Lyapunov function:

$$\begin{aligned}V_t(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) &\equiv V_0(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \\ &+ V_I(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus})\end{aligned}, \qquad (22)$$

where $V_0(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \equiv V_0(\mathbf{s},\boldsymbol{\omega})$. In the absence of disturbances, the derivative of this function evaluated along the trajectories using $\mathbf{f}_{\mathbf{t}}(\mathbf{s},\boldsymbol{\omega},\hat{\mathbf{I}})$ becomes:

$$\begin{aligned}W_t(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_t(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \\ &= W_{t0}(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \\ &+ W_{tI}(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus})\end{aligned}, \qquad (23)$$

where

$$\begin{aligned}W_{t0}(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_0(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \\ &= -\boldsymbol{\omega}^{\mathbf{T}}\mathbf{K}\boldsymbol{\omega} \\ &+ \boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Omega}^{\times}\Delta\mathbf{I}\boldsymbol{\Omega} \\ &+ \boldsymbol{\omega}^{\mathbf{T}}\Delta\overline{\mathbf{H}}\end{aligned}, \qquad (24)$$

$$\begin{aligned}W_{tI}(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_I(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) \\ &= \Delta\dot{\mathbf{I}}^{\oplus\mathbf{T}}\boldsymbol{\Gamma}\Delta\mathbf{I}^{\oplus} \\ &= \dot{\hat{\mathbf{I}}}^{\oplus\mathbf{T}}\boldsymbol{\Gamma}\Delta\mathbf{I}^{\oplus}\end{aligned} \qquad (25)$$

and $\Delta\overline{\mathbf{H}} \equiv \Delta\mathbf{I}\left[\mathbf{C}(\mathbf{s})\dot{\overline{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\boldsymbol{\Omega}\right]$. Furthermore,

$$\begin{aligned}W_t(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) &= -\boldsymbol{\omega}^{\mathbf{T}}\mathbf{K}\boldsymbol{\omega} \\ &+ \boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Omega}^{\times}\boldsymbol{\Omega}^{\otimes}\Delta\mathbf{I}^{\oplus} \\ &+ \boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Pi}^{\otimes}\Delta\mathbf{I}^{\oplus} \\ &+ \dot{\hat{\mathbf{I}}}^{\oplus\mathbf{T}}\boldsymbol{\Gamma}\Delta\mathbf{I}^{\oplus}\end{aligned}, \qquad (26)$$

where $\boldsymbol{\Pi} \equiv \mathbf{C}(\mathbf{s})\dot{\overline{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\boldsymbol{\Omega}$. Clearly, using the following adaptation law

$$\dot{\hat{\mathbf{I}}}^{\oplus} = -\boldsymbol{\Gamma}^{-1}\left[\boldsymbol{\Omega}^{\times}\boldsymbol{\Omega}^{\otimes} + \boldsymbol{\Pi}^{\otimes}\right]^{\mathbf{T}}\boldsymbol{\omega} \qquad (27)$$

leads to a negative semi-definite derivative:

$$W_t(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) = -\boldsymbol{\omega}^{\mathbf{T}}\mathbf{K}\boldsymbol{\omega} \le 0. \qquad (28)$$

Application of LaSalle's invariance principle is similar to the non-adaptive case. All trajectories are bounded, $\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I} \in \ell_{\infty}$, and in the set $\mathrm{N} = \left\{(\mathbf{s},\boldsymbol{\omega},\Delta\mathbf{I}^{\oplus}) : W_t = 0\right\}$ must have $\boldsymbol{\omega} = \mathbf{0}$, which leads to $\dot{\mathbf{s}} = \mathbf{0}$ based on Eq.(10) assuming $\mathbf{p}(\mathbf{s}) \in \ell_{\infty}$ and to $\dot{\hat{\mathbf{I}}} = \mathbf{0}$ based on Eq.(27). Similarly, considering higher derivatives gives $\dot{\boldsymbol{\omega}} = \mathbf{0}$. However, the solution now includes potentially non-zero terms based on Eqs.(11,12,21), which means that the attitude error in the absence of disturbances must satisfy:

$$\begin{aligned}\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) &= -\Delta\mathbf{I}_{\infty}\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}^{\times}\Delta\mathbf{I}_{\infty}\boldsymbol{\Omega} \\ &= \mathbf{const}\end{aligned}. \qquad (29)$$

In other words, $\mathbf{s}$ may not approach $\mathbf{1}_{\mathbf{s}}$ if the reference trajectory can be asymptotically modeled as that of the rigid body driven by a non-zero constant torque with constant symmetric but not necessarily positive definite matrix $\Delta\mathbf{I}_{\infty} = \hat{\mathbf{I}}_{\infty} - \mathbf{I} \ne \mathbf{0}$. One simple example of such a trajectory is a slew with constant acceleration or deceleration. Then having $\Delta\mathbf{I}_{\infty} = k\mathbf{E}_{\mathbf{3}}$ with $k \ne 0$ leads to

$$\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) = -k\dot{\boldsymbol{\Omega}} = \mathbf{const} \ne \mathbf{0} \qquad (30)$$

and $\mathbf{s} \ne \mathbf{1}_{\mathbf{s}}$. However, it is clear that inertially fixed reference trajectories lead to $\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) = \mathbf{0}$ and $\mathbf{s} = \mathbf{1}_{\mathbf{s}}$. The result is not surprising, as control laws exist for attitude stabilization that do not require knowledge of the inertia matrix. This is known as the *reduction* property of attitude stabilization. It is also intuitively clear that sufficiently excited reference trajectories would require $\mathbf{\Delta I}_{\infty} = \mathbf{0}$ in order to satisfy Eq.(29). This will also lead to $\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) = \mathbf{0}$ and $\mathbf{s} = \mathbf{1}_{\mathbf{s}}$. This result is well known in estimation theory as *persistence of excitation*. Therefore, in general, the largest invariant set in N is the set $\mathrm{M} = \{(\mathbf{s}, \boldsymbol{\omega}, \mathbf{\Delta I}) : \mathbf{s} = \mathbf{s}_{\infty}, \boldsymbol{\omega} = \mathbf{0}, \mathbf{\Delta I} = \mathbf{\Delta I}_{\infty}\}$, where $\mathbf{s}_{\infty} = \mathbf{const}$ and $\mathbf{\Delta I}_{\infty} = \mathbf{const}$. If the reference trajectory is inertially fixed, $\mathbf{s}_{\infty} = \mathbf{1}_{\mathbf{s}}$. If the reference trajectory is sufficiently excited, both $\mathbf{s}_{\infty} = \mathbf{1}_{\mathbf{s}}$ and $\mathbf{\Delta I}_{\infty} = \mathbf{0}$. According to LaSalle's invariance principle the system governed by Eqs.(10,11,12,21) is globally asymptotically stable. In particular, all trajectories of the system asymptotically approach M, i.e.

$$\lim_{t\to\infty} \mathbf{s(t)} = \mathbf{s}_{\infty}, \lim_{t\to\infty} \boldsymbol{\omega}\mathbf{(t)} = \mathbf{0}, \lim_{t\to\infty} \mathbf{\Delta I(t)} = \mathbf{\Delta I}_{\infty}.$$

ADAPTIVE TRACKING WITH MOMENTUM BIAS AND NO ANGULAR VELOCITY MEASUREMENTS

The passivity and cascade nature of rigid body attitude dynamics and kinematics have been used successfully to replace angular velocity measurements with a lead filter. In particular angular velocity free control laws have been developed using quaternions for stabilization[3] and Modified Rodrigues parameters (MRPs) for tracking[7]. Note that all of them require continuous time-varying external torque for proper cancellation of the angular velocity dependent terms in stability analysis. Momentum exchange devices are better suited for this mode of operation then pulsed jets. However, the momentum bias becomes coupled with the angular velocity and in the case of tracking remains coupled even when the relative angular velocity approaches zero. In the presence of the angular velocity measurements, this coupling can be easily cancelled out in the formulation of the control law as can be seen from the development in the previous section. Without the angular velocity measurements, the control law must be modified differently so that it remains angular velocity free. In order to take advantage of the passivity via LaSalle's invariance principle, a lead filter similar to those that have proposed for rigid-body without momentum bias is introduced. The filter includes additional stable dynamics governed by the following differential equations:

$$\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{b(s)}, \tag{31}$$

where $\mathbf{A}$ is any Hurwitz matrix, i.e.

$$\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{PA} = -\mathbf{Q}, \tag{32}$$

with $\mathbf{P} = \mathbf{P}^{\mathbf{T}} > 0$ and $\mathbf{Q} = \mathbf{Q}^{\mathbf{T}} > 0$; $\mathbf{b(s)}$ is the attitude representation dependent function that satisfies the following criterion: $\dot{\mathbf{b}}\mathbf{(s)} = \mathbf{0}$ necessarily implies $\dot{\mathbf{s}} = \mathbf{0}$. The kinematics Eq.(10) remains the same, but the dynamics Eq.(11) is modified to include a different control law formulation $\bar{\mathbf{f}}_{\mathbf{p}}(\mathbf{s}, \mathbf{z})$:

$$\begin{aligned} \mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\Omega}^{\times}[\mathbf{I}\boldsymbol{\Omega} + \mathbf{h}] - \overline{\mathbf{H}} \\ + \bar{\mathbf{f}}_{\mathbf{p}}(\mathbf{s}, \mathbf{z}) + \mathbf{d}(\mathbf{S}, \boldsymbol{\Omega}) \end{aligned}. \tag{33}$$

This control law, $\bar{\mathbf{f}}_{\mathbf{p}}(\mathbf{s}, \mathbf{z})$, does not depend on the angular velocity $\boldsymbol{\omega}$:

$$\begin{aligned} \bar{\mathbf{f}}_{\mathbf{p}}(\mathbf{s}, \mathbf{z}) \equiv -\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) \\ - \mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{b}}(\mathbf{s})\mathbf{P}\dot{\mathbf{z}} \\ + \mathbf{H}_{\mathbf{p}} + \boldsymbol{\eta}^{\times}\mathbf{h} \end{aligned}, \tag{34}$$

where

$$\mathbf{H}_{\mathbf{p}} \equiv \mathbf{IC(s)}\dot{\overline{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\mathbf{I}\boldsymbol{\eta} \tag{35}$$

and $\boldsymbol{\lambda}_{\mathbf{b}}(\mathbf{s}) \equiv \left[\dfrac{\partial \mathbf{b(s)}}{\partial \mathbf{s}}\right]^{\mathbf{T}}$.

Stability under this new control law formulation can be demonstrated by using the original Lyapunov function $V_0(\mathbf{s}, \boldsymbol{\omega})$ described by Eq.(14) and extending it to include terms depending on the filter variables:

$$V_p(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \equiv V_0(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) + V_a(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}), \tag{36}$$

where $V_0(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \equiv V_0(\mathbf{s}, \boldsymbol{\omega})$ and

$$V_a(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \equiv \frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{P}\dot{\mathbf{z}}. \tag{37}$$

The derivative the Lyapunov function $V_p(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z})$ evaluated along the trajectories using $\bar{\mathbf{f}}_{\mathbf{p}}(\mathbf{s}, \mathbf{z})$ becomes:

$$\begin{aligned} W_p(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \equiv \dot{V}_p(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) = W_{p0}(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \\ + W_{pa}(\mathbf{s}, \boldsymbol{\omega}, \mathbf{z}) \end{aligned},$$

where

$$W_{p0}(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) \equiv \dot{V}_0(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) = -\boldsymbol{\omega}^{\mathbf{T}}\left(\boldsymbol{\Omega}^{\times}\left[\mathbf{I}\boldsymbol{\Omega}+\mathbf{h}\right]+\overline{\mathbf{H}}\right) + \boldsymbol{\omega}^{\mathbf{T}}\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) + \boldsymbol{\omega}^{\mathbf{T}}\mathbf{f}_{\mathbf{p}}(\mathbf{s},\mathbf{z}), \quad (39)$$

$$\begin{aligned} W_{pa}(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) &\equiv \dot{V}_a(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) \\ &= \frac{1}{2}\left\{\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{P}\ddot{\mathbf{z}} + \ddot{\mathbf{z}}^{\mathbf{T}}\mathbf{P}\dot{\mathbf{z}}\right\} \\ &= \frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{P}\left[\mathbf{A}\dot{\mathbf{z}} + \boldsymbol{\lambda}_{\mathbf{b}}^{\mathbf{T}}(\mathbf{s})\dot{\mathbf{s}}\right] + \frac{1}{2}\left[\mathbf{A}\dot{\mathbf{z}} + \boldsymbol{\lambda}_{\mathbf{b}}^{\mathbf{T}}(\mathbf{s})\dot{\mathbf{s}}\right]^{\mathbf{T}}\mathbf{P}\dot{\mathbf{z}} \\ &= -\frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{z}} + \boldsymbol{\omega}^{\mathbf{T}}\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{b}}(\mathbf{s})\mathbf{P}\dot{\mathbf{z}} \end{aligned} . \quad (40)$$

Hence, the derivative simplifies to

$$W_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) = -\frac{\mathbf{1}}{\mathbf{2}}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{z}} + \boldsymbol{\Delta}, \quad (41)$$

where

$$\begin{aligned} \boldsymbol{\Delta} &\equiv \boldsymbol{\omega}^{\mathbf{T}}\left(\mathbf{H}_{\mathbf{p}} - \overline{\mathbf{H}} - \boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\Omega}\right) + \boldsymbol{\omega}^{\mathbf{T}}\left(\boldsymbol{\eta}^{\times}\mathbf{h} - \boldsymbol{\Omega}^{\times}\mathbf{h}\right) \\ &= \boldsymbol{\omega}^{\mathbf{T}}\left[\boldsymbol{\eta}^{\times}\mathbf{I}\boldsymbol{\eta} - \mathbf{I}\boldsymbol{\eta}^{\times}\boldsymbol{\Omega} - \boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\Omega}\right] - \boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\omega}^{\times}\mathbf{h} \\ &= \boldsymbol{\omega}^{\mathbf{T}}\left[-\boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}\left(\boldsymbol{\omega}^{\times}\boldsymbol{\Omega}\right)\right] \\ &= -\boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \left(\mathbf{I}\boldsymbol{\omega}\right)^{\mathbf{T}}\left(\boldsymbol{\omega}^{\times}\boldsymbol{\Omega}\right) \\ &= -\boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\omega}^{\mathbf{T}}\boldsymbol{\Omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \mathbf{0} \end{aligned} . \quad (42)$$

This means that

$$W_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) = -\frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{z}} \le 0 . \quad (43)$$

Closely following the analysis in the absence of momentum bias,[7] global asymptotic stability can be shown based on LaSalle's invariance principle. As $V_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z})$ is radially unbounded and $W_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) \le 0$, all solutions are bounded. The set $\mathrm{N} = \{(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) : W_p = 0\}$ contains trajectories with $\dot{\mathbf{z}} = \mathbf{0}$, which leads to $\dot{\mathbf{b}}(\mathbf{s}) = \mathbf{0}$ based on Eq.(31), which, in turn, leads to $\dot{\mathbf{s}} = \mathbf{0}$. Then, from Eq.(10), $\boldsymbol{\omega} = \mathbf{0}$ provided that $\mathbf{p}(\mathbf{s})$ is non-singular. Furthermore, as higher derivatives of $\mathbf{s}$ and $\mathbf{z}$ are all zero trajectories that belong to the same set, $\dot{\boldsymbol{\omega}} = \mathbf{0}$, which leads to $\mathbf{s} = \mathbf{1}_{\mathbf{s}}$ based on Eqs.(33,34). Finally, based on Eq.(31), steady state of the filter becomes $\mathbf{z} = \mathbf{z}_{\infty} \equiv -\mathbf{A}^{-1}\mathbf{b}(\mathbf{1}_{\mathbf{s}})$. The largest invariant set in N is thus the set $\mathrm{M} = \{(\mathbf{s},\boldsymbol{\omega},\mathbf{z}) : \mathbf{s} = \mathbf{1}_{\mathbf{s}}, \boldsymbol{\omega} = \mathbf{0}, \mathbf{z} = \mathbf{z}_{\infty}\}$. According to LaSalle's invariance principle the system governed by Eqs.(10,31,33,34) is globally asymptotically stable. In particular, all trajectories of the system asymptotically approach M, i.e.

$$\lim_{t\to\infty}\mathbf{s(t)} = \mathbf{1}_{\mathbf{s}}, \lim_{t\to\infty}\boldsymbol{\omega}\mathbf{(t)} = \mathbf{0}, \lim_{t\to\infty}\mathbf{z(t)} = \mathbf{z}_{\infty}.$$

This stability result is particularly interesting because it shows that momentum exchange devices can be used for angular velocity free attitude tracking and that the effect on stability of the coupling between the momentum bias and the angular velocity can be cancelled by including the coupling of the momentum bias and the *desired* angular velocity in the control law formulation.

Quaternion only tracking with momentum bias

Results presented above in this section are independent from a specific attitude representation and, thus, can be used to generate different control laws. For example, if the unit quaternion representation is selected

$$\mathbf{q} \equiv \left[\mathbf{q}_{\mathbf{V}} \quad q_4\right]^{\mathbf{T}}, \quad (44)$$

where $\|\mathbf{q}\| = 1$ and $\mathbf{1}_{\mathbf{q}} = \left[\mathbf{0} \quad 1\right]^{\mathbf{T}}$ with

$$\mathbf{q}_{\mathbf{V}} \equiv \left[q_1 \quad q_2 \quad q_3\right]^{\mathbf{T}}, \quad (45)$$

the direction cosine matrix becomes defined as

$$\mathbf{C(q)} \equiv \left(q_4^2 - \mathbf{q}_{\mathbf{V}}^{\mathbf{T}}\mathbf{q}_{\mathbf{V}}\right)\mathbf{E}_{\mathbf{3}} + 2\mathbf{q}_{\mathbf{V}}\mathbf{q}_{\mathbf{V}}^{\mathbf{T}} - 2q_4\mathbf{q}_{\mathbf{V}}^{\times} \quad (46)$$

and kinematics uses

$$\mathbf{p(q)} \equiv \frac{1}{2}\begin{bmatrix} q_4\mathbf{E}_{\mathbf{3}} + \mathbf{q}_{\mathbf{V}}^{\times} \\ -\mathbf{q}_{\mathbf{V}}^{\mathbf{T}} \end{bmatrix}. \quad (47)$$

The following Lyapunov function is often used in design of quaternion based stabilization control laws:

$$V_s(\mathbf{q}) \equiv \frac{k_q}{2}\left(\mathbf{q}_\mathrm{V}^\mathrm{T}\mathbf{q}_\mathrm{V} + [1-q_4]^2\right), \quad (48)$$

where $k_q > 0$. The augmented dynamics can then be driven by the following function of attitude

$$\mathbf{b}(\mathbf{q}) \equiv k_z \mathbf{q}_\mathrm{V}, \quad (49)$$

where $k_z > 0$, which clearly satisfies the requirement that $\dot{\mathbf{b}}(\mathbf{q}) = \mathbf{0}$ implies $\dot{\mathbf{q}}_\mathrm{V} = \mathbf{0}$ and $\dot{\mathbf{q}} = \mathbf{0}$.

The resulting control law can now be built using

$$\boldsymbol{\lambda}_\mathrm{V}(\mathbf{q}) \equiv k_q[\mathbf{q}_\mathrm{V} \quad (q_4 - 1)]^\mathrm{T} \quad (50)$$

and

$$\boldsymbol{\lambda}_\mathbf{b}(\mathbf{q}) \equiv k_z[\mathbf{E}_3 \quad \mathbf{0}_{3\times 1}]^\mathrm{T}, \quad (51)$$

which leads to

$$\begin{aligned}\bar{\mathbf{f}}_\mathbf{p}(\mathbf{q},\mathbf{z}) &= -\mathbf{p}^\mathrm{T}(\mathbf{q})\boldsymbol{\lambda}_\mathrm{V}(\mathbf{q}) \\ &\quad - \mathbf{p}^\mathrm{T}(\mathbf{q})\boldsymbol{\lambda}_\mathbf{b}(\mathbf{q})\mathbf{P}\dot{\mathbf{z}} \\ &\quad + \mathbf{H}_\mathbf{p} + \boldsymbol{\eta}^\times \mathbf{h} \\ &= -\frac{k_q}{2}\mathbf{q}_\mathrm{V} - \frac{k_z}{2}q_4\mathbf{P}\dot{\mathbf{z}} \\ &\quad + \frac{k_z}{2}\mathbf{q}_\mathrm{V}^\times \mathbf{P}\dot{\mathbf{z}} + \mathbf{H}_\mathbf{p} + \boldsymbol{\eta}^\times \mathbf{h}\end{aligned}, \quad (52)$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + k_z \mathbf{q}_\mathrm{V}. \quad (53)$$

Since this control law was generated using generic framework developed in this section, it is necessarily angular velocity free and globally asymptotically stable with respect to the reference attitude trajectory. Also, since according to Eq.(49) $\mathbf{b}(\mathbf{1}_\mathbf{q}) = \mathbf{0}$, the filter state will asymptotically approach $\mathbf{z}_\infty = \mathbf{0}$.

Making angular velocity free control adaptive presents interesting challenges, because the adaptation law itself also must be angular velocity free. It was shown in the previous section that the straightforward application of MRAC principles produces the adaptation law (Eq.(27)), which includes the angular velocity. An equivalent formulation of this adaptation law have been proposed recently.[10] This formulation uses MRPs for attitude representation and is based on the integration by parts using the separation of MRPs from the reference trajectory. The results presented below follow the same approach, but without using a particular attitude representation.

The original Lyapunov function $V_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z})$ is extended with the same terms that were used in the previous section for adaptive control design:

$$\begin{aligned}V_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^\oplus) &\equiv V_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^\oplus) \\ &\quad + V_I(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^\oplus)\end{aligned}, \quad (54)$$

where $V_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^\oplus) \equiv V_p(\mathbf{s},\boldsymbol{\omega},\mathbf{z})$ and $V_I(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^\oplus) \equiv V_I(\mathbf{s},\boldsymbol{\omega},\mathbf{z})$. Following the same MRAC principles that were used in the previous section, the adaptive control is designed by replacing the true inertia matrix with its estimate:

$$\begin{aligned}\bar{\mathbf{f}}_\mathbf{pt}(\mathbf{s},\mathbf{z},\hat{\mathbf{I}}) &\equiv -\mathbf{p}^\mathrm{T}(\mathbf{s})\boldsymbol{\lambda}_\mathrm{V}(\mathbf{s}) \\ &\quad - \mathbf{p}^\mathrm{T}(\mathbf{s})\boldsymbol{\lambda}_\mathbf{b}(\mathbf{s})\mathbf{P}\dot{\mathbf{z}} \\ &\quad + \hat{\mathbf{H}}_\mathbf{p} + \boldsymbol{\eta}^\times \mathbf{h}\end{aligned}, \quad (55)$$

where

$$\begin{aligned}\hat{\mathbf{H}}_\mathbf{p} &\equiv \hat{\mathbf{I}}\mathbf{C}(\mathbf{s})\dot{\bar{\boldsymbol{\omega}}} + \boldsymbol{\eta}^\times \hat{\mathbf{I}}\boldsymbol{\eta} \\ &= [\bar{\mathbf{G}}_\mathbf{a} + \bar{\mathbf{G}}_\mathbf{r}]\hat{\mathbf{I}}^\oplus\end{aligned}, \quad (56)$$

$$\bar{\mathbf{G}}_\mathbf{a} \equiv \sum_{i=1}^{3}\mathbf{C}_\mathbf{i}^\otimes \dot{\bar{\omega}}_i, \quad (57)$$

$$\bar{\mathbf{G}}_\mathbf{r} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{C}_\mathbf{i}^\times \mathbf{C}_\mathbf{j}^\otimes \bar{\omega}_i\bar{\omega}_j. \quad (58)$$

Here, $\mathbf{C}_\mathbf{k} \in \Re^3, k = 1,2,3$ are the columns of the direction cosine matrix

$$\mathbf{C}(\mathbf{s}) = [\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3] \quad (59)$$

and $\bar{\omega}_k, \dot{\bar{\omega}}_k \in \Re, k = 1,2,3$ are the components of the reference angular velocity and angular acceleration computed in the reference frame. The adaptation law designed can now be written in the form that separates the attitude dependent variables from the reference trajectory dependent variables:

$$\begin{aligned}\dot{\hat{\mathbf{I}}} &= -\boldsymbol{\Gamma}[\bar{\mathbf{G}}_\mathbf{a} + \bar{\mathbf{G}}_\mathbf{r}]^\mathrm{T}\boldsymbol{\omega} \\ &= -\boldsymbol{\Gamma}[\bar{\mathbf{G}}_\mathbf{a} + \bar{\mathbf{G}}_\mathbf{r}]^\mathrm{T}\mathbf{p}^{-1}(\mathbf{s})\dot{\mathbf{s}} \\ &= -\boldsymbol{\Gamma}\sum_{i=1}^{3}\dot{\bar{\omega}}_i\mathbf{F}_\mathbf{i}''(\mathbf{s})\dot{\mathbf{s}} \\ &\quad - \boldsymbol{\Gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\bar{\omega}_i\bar{\omega}_j\mathbf{F}_\mathbf{ij}'(\mathbf{s})\dot{\mathbf{s}}\end{aligned}, \quad (60)$$

where

$$\mathbf{F}''_{\mathbf{i}}(\mathbf{s}) \equiv \left[\mathbf{C}_{\mathbf{i}}^{\otimes}\right]^{\mathbf{T}} \mathbf{p}^{-1}(\mathbf{s}) \tag{61}$$

and

$$\mathbf{F}'_{\mathbf{ij}}(\mathbf{s}) = \left[\mathbf{C}_{\mathbf{i}}^{\times}\mathbf{C}_{\mathbf{j}}^{\otimes}\right]^{\mathbf{T}} \mathbf{p}^{-1}(\mathbf{s}) \tag{62}$$

are matrices of the appropriate dimensions and it is assumed that $\mathbf{p}(\mathbf{s})$ is non-singular. Integration of both left- and right-hand sides of Eq.(60) yields an equivalent form of the adaptation law:

$$\begin{aligned}\hat{\mathbf{I}}(t) = \hat{\mathbf{I}}(t_0) &+ \\ &-\mathbf{\Gamma}\sum_{i=1}^{3}\dot{\bar{\omega}}_i \int_{\mathbf{s}(t_0)}^{\mathbf{s}(t)} \mathbf{F}''_{\mathbf{i}}(\mathbf{s})d\mathbf{s} \\ &+\mathbf{\Gamma}\sum_{i=1}^{3}\int_{t_0}^{t}\ddot{\bar{\omega}}_i \int_{\mathbf{s}(t_0)}^{\mathbf{s}(\tau)} \mathbf{F}''_{\mathbf{i}}(\mathbf{s})d\mathbf{s}d\tau \\ &-\mathbf{\Gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\bar{\omega}_i\bar{\omega}_j \int_{\mathbf{s}(t_0)}^{\mathbf{s}(t)} \mathbf{F}'_{\mathbf{ij}}(\mathbf{s})d\mathbf{s} \\ &+\mathbf{\Gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{t_0}^{t}\dot{\bar{\omega}}_i\bar{\omega}_j \int_{\mathbf{s}(t_0)}^{\mathbf{s}(\tau)} \mathbf{F}'_{\mathbf{ij}}(\mathbf{s})d\mathbf{s}d\tau \\ &+\mathbf{\Gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{t_0}^{t}\bar{\omega}_i\dot{\bar{\omega}}_j \int_{\mathbf{s}(t_0)}^{\mathbf{s}(\tau)} \mathbf{F}'_{\mathbf{ij}}(\mathbf{s})d\mathbf{s}d\tau\end{aligned} \quad . \tag{63}$$

Note that the integration by parts that was used to integrate right hand side of Eq.(60) requires integration of a matrix with respect to a vector, which results in another vector of a different dimension. Also, note that this adaptation law, while equivalent to Eq.(60), is angular velocity free: it only uses the relative attitude, its integrals and the angular velocity and acceleration of the reference trajectory.

The stability of the closed-loop system using the control law Eq.(55) and the adaptation law Eq.(60) can be shown by differentiating the extended Lyapunov function $V_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus})$ along the closed-loop trajectories:

$$\begin{aligned}W_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) \\ &= W_{ptp}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) \\ &+ W_{ptI}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus})\end{aligned} , \tag{64}$$

where

Hence, the derivative becomes

$$\begin{aligned}W_{ptp}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_{p}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) \\ &= -\frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{z}} + \Delta\mathbf{H}_{\mathbf{p}}\end{aligned} , \tag{65}$$

with

$$\begin{aligned}\Delta\mathbf{H}_{\mathbf{p}} &\equiv \Delta\mathbf{I}\mathbf{C}(\mathbf{s})\dot{\bar{\boldsymbol{\omega}}} + \boldsymbol{\eta}^{\times}\Delta\mathbf{I}\boldsymbol{\eta} \\ &= \left[\overline{\mathbf{G}}_{\mathbf{a}} + \overline{\mathbf{G}}_{\mathbf{r}}\right]\Delta\mathbf{I}^{\oplus}\end{aligned} \tag{66}$$

and

$$\begin{aligned}W_{ptI}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) &\equiv \dot{V}_{I}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) \\ &= \dot{\hat{\mathbf{I}}}^{\oplus\mathbf{T}}\mathbf{\Gamma}\Delta\mathbf{I}^{\oplus}\end{aligned} \quad . \tag{67}$$

Combining the results of Eqs.(65-67) yields:

$$W_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) = -\frac{1}{2}\dot{\mathbf{z}}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{z}} \le 0 \quad . \tag{68}$$

Once again global asymptotic stability can be shown based on LaSalle's invariance principle. As $V_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus})$ is radially unbounded and $W_{pt}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) \le 0$, all solutions are bounded. The set $\mathrm{N} = \left\{(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}^{\oplus}) : W_{pt} = 0\right\}$ contains trajectories with $\dot{\mathbf{z}} = \mathbf{0}$, which leads to $\dot{\mathbf{b}}(\mathbf{s}) = \mathbf{0}$ and $\dot{\mathbf{s}} = \mathbf{0}$ based on Eq.(31). Then, from Eq.(10), $\boldsymbol{\omega} = \mathbf{0}$, provided that $\mathbf{p}(\mathbf{s})$ is non-singular and, from Eq.(60), $\Delta\dot{\mathbf{I}} = \mathbf{0}$. Furthermore, as higher derivatives of $\mathbf{s}$ and $\mathbf{z}$ are all zero trajectories that belong to the same set, it follows that $\dot{\boldsymbol{\omega}} = \mathbf{0}$, which yields the following equation:

$$\begin{aligned}\mathbf{p}^{\mathbf{T}}(\mathbf{s})\boldsymbol{\lambda}_{\mathbf{V}}(\mathbf{s}) &= -\Delta\mathbf{I}\dot{\mathbf{\Omega}} - \mathbf{\Omega}^{\times}\Delta\mathbf{I}\mathbf{\Omega} \\ &= \mathbf{const}\end{aligned} \quad . \tag{69}$$

This equation is identical to Eq.(29) derived in the previous section, so the same analysis applies. The largest invariant set in N is the set $\mathrm{M} = \left\{\begin{matrix}(\mathbf{s},\boldsymbol{\omega},\mathbf{z},\Delta\mathbf{I}) : \mathbf{s} = \mathbf{s}_{\infty}, \boldsymbol{\omega} = \mathbf{0}, \\ \mathbf{z} = \mathbf{z}_{\infty}, \Delta\mathbf{I} = \Delta\mathbf{I}_{\infty}\end{matrix}\right\}$

with $\mathbf{s}_{\infty} = \mathbf{const}$, $\Delta\mathbf{I}_{\infty} = \mathbf{const}$ and $\mathbf{z}_{\infty} = -\mathbf{A}^{-1}\mathbf{b}(\mathbf{s}_{\infty})$. If the reference trajectory is inertially fixed, $\mathbf{s}_\infty = \mathbf{1}_\mathbf{s}$. If the reference trajectory is sufficiently excited, both $\mathbf{s}_\infty = \mathbf{1}_\mathbf{s}$ and $\Delta\mathbf{I}_\infty = \mathbf{0}$. According to LaSalle's invariance principle the system governed by Eqs.(10, 31, 33, 55, 63) is globally asymptotically stable. In particular, all trajectories of the system asymptotically approach M, i.e.

$$\lim_{t\to\infty}\mathbf{s(t)} = \mathbf{s}_\infty, \lim_{t\to\infty}\boldsymbol{\omega}\mathbf{(t)} = \mathbf{0}, \lim_{t\to\infty}\mathbf{z(t)} = \mathbf{z}_\infty,$$
$$\lim_{t\to\infty}\Delta\mathbf{I(t)} = \Delta\mathbf{I}_\infty .$$

SIMULATION

A simulation included in this paper has been performed using attitude simulation and visualization capabilities of Satellite Toolkit (STK) and MATLAB. The dynamics of a rigid body with momentum bias and the quaternion kinematics have been defined and numerically integrated in STK. The actual control laws have been implemented as MATLAB functions callable from STK. 3D visualizations have been created in STK and plots have been created in MATLAB.

The simulation demonstrates performance of the non-adaptive quaternion only control law described in this paper (Eqs.(52,53)). The reference trajectory has been designed in STK to follow nadir pointing attitude until a target missile is detected over the horizon, at which point the spacecraft switches to target pointing. Once the missile is no longer visible, the spacecraft is returned to follow nadir pointing attitude. Clearly, the angular velocity and acceleration along the reference trajectory are not zero and depend on the relative motion of the spacecraft and its target. The magnitude of angular velocity and acceleration along the reference trajectory are shown in Figs.1,2. The following parameters have been selected for the control law: $\mathbf{A} = -0.1\mathbf{E_3}$, $\mathbf{P} = 200\mathbf{E_3}$, $k_q = 50$, $k_z = 1$ for the spacecraft inertia matrix of

$$\mathbf{I} = \begin{bmatrix} 21.4 & 2.1 & 1.8 \\ 2.1 & 20.1 & 0.5 \\ 1.8 & 0.5 & 5 \end{bmatrix} 10^3 (kgm^2) .$$

The tracking performance is evaluated using 3D visualizations and time history plots, which are included below. The plots depict magnitudes of the angular velocity error and of the attitude error (Figs.3,4) as well as magnitudes of the filter state and of the momentum bias (Figs.5,6). All indicate successful performance of the quaternion only feedback using momentum exchange devices.

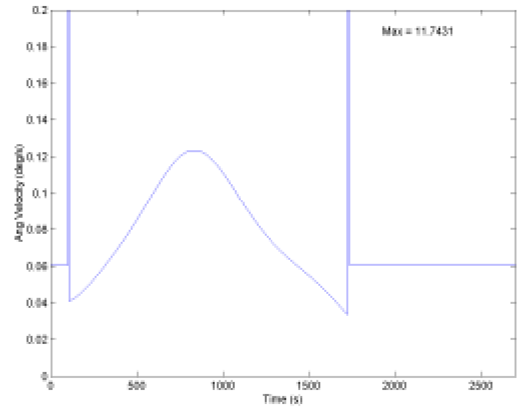

Figure 1 Reference angular velocity

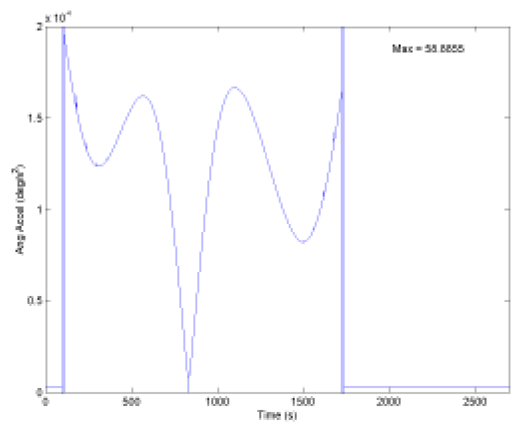

Figure 2 Reference angular acceleration

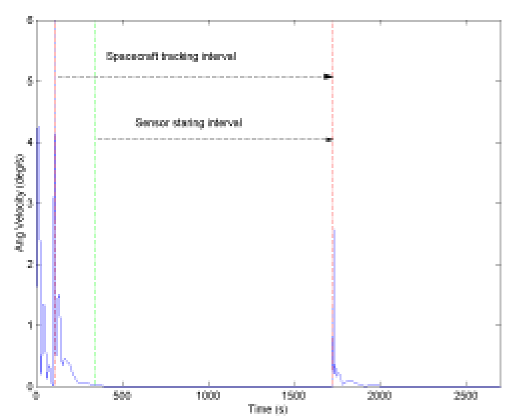

Figure 3 Angular velocity error

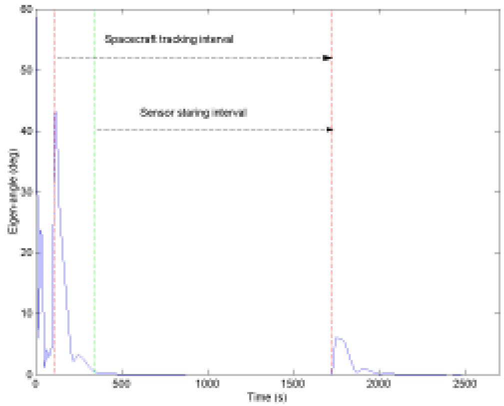

Figure 4 Attitude error

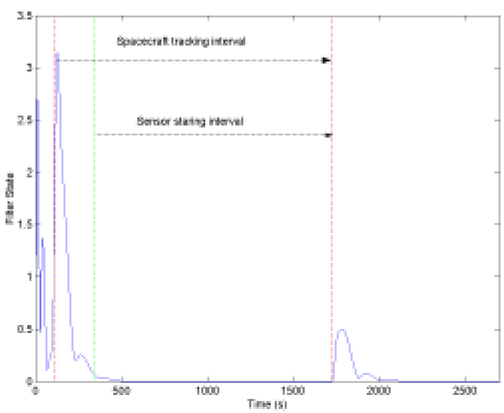

Figure 5 Filter state

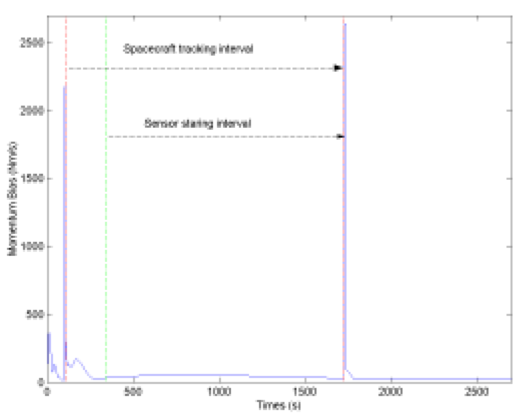

Figure 6 Momentum bias

CONCLUSIONS

A category of passivity based attitude tracking control laws has been considered. In their original form, these control laws do not include momentum bias. This paper has generalized them to include momentum bias with and without angular velocity measurements. Also, MRAC extensions of these control laws for use with the unknown constant inertia matrix have been proposed and evaluated. The results have indicated that it is possible to successfully track agile reference trajectories using momentum exchange devices even in the absence of angular velocity measurements. At the same time, it is possible that MRAC extension of attitude tracking will result in the constant steady-state attitude error.

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