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Correct Modeling of the Indirect Term for Third-Body Perturbations

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The indirect term in the formula for third body perturbations models the acceleration of the primary body due to the third body. This term is necessary because the integration frame, which has its origin at the center of the primary body, is not inertial. The term is normally computed analytically, assuming both bodies are point masses and only gravitational forces affect the primary body. However, these assumptions lead to inaccuracies when other forces act on the primary body.

INTRODUCTION

When using numerical integration to compute the trajectory of a spacecraft, the classical formula for modeling third-body perturbations (Eq. (6) derived below) contains two terms: a direct term and an indirect term. The direct term models the effect of the third body on the spacecraft, and the indirect term models the effect of the third body on the primary body. The indirect term is necessary because the integration frame, which has its origin at the center of mass of the primary body, is not inertial.

The indirect term in the classical formula assumes that the only forces acting on the primary body are the point mass effects of the third bodies included in the integration. If the planetary ephemeris used in the integration comes from an *n*-body integration where only point-mass effects were included, this assumption is correct. However, if the planetary ephemeris comes from another source, such as JPL DE 405 or JPL Spice files, other terms exist in the acceleration of the primary body which are ignored. Ignoring these terms can lead to inaccuracies in the computed trajectory.

Instead of using the classical formula, the acceleration of the primary body can be found by numerically differentiating the planetary ephemeris. This approach should yield better accuracy than the classical approach because it does not ignore any terms. Alternatively,

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if the integration is performed in an inertial frame, such as one with the origin at the solar system barycenter, the indirect term is not needed.

In this paper, we use three different formulations of the equations of motions to examine the effects of modeling third body gravitational forces on a spacecraft that is orbiting near a celestial body. Tests are performed in the Earth-Moon system as well as about moons in the Jupiter and Saturn systems. We wish to determine whether the choice of formulation or reference frame has any consequences for obtaining an accurate trajectory. While in principle neither formulation nor frame should affect the resulting trajectory, in practice both choices do.

EQUATIONS OF MOTION

The derivation of the equations of motion for a spacecraft traveling in the solar system starts from the choice of a suitable inertial frame. The axes of the inertial frame can be taken as J2000 axes (or the ICRF axes if J2000 axes do not suffice). The origin of the inertial frame is taken as the solar system barycenter. Though the solar system barycenter defines the inertial origin, it is rarely used as the origin for the frame in which the equations of motions themselves will be expressed. It is much more common to use the position of a celestial body in the solar system as the origin for the equations of motion. Earth-orbiting spacecraft typically use the Earth as the origin. Of course, once the inertial frame is determined, the equations of motion written for the inertial origin can be translated to any origin by simply including the effects of the origin's acceleration. In Newtonian mechanics, the motion of the spacecraft is independent of the choice of origin that locates the spacecraft if the same inertial frame is being used to generate the equations.

When modeling the force environment of the system, forces on the spacecraft arising from celestial bodies are considered while forces on the celestial bodies arising from the spacecraft are ignored. Thus, the motion of the celestial bodies can be determined independently from the motion of the spacecraft. Typical methods for modeling the motion of celestial bodies include (i) analytic approximations (e.g., Brouwer's formula for the location of the Moon) and (ii) interpolation of a numerical integration of an N-body problem (e.g., the Developmental Ephemeris created by JPL (Ref. 1)).

Forces on the spacecraft may include gravity (arising from a specified set of celestial bodies), tides, solar radiation pressure, atmospheric drag, thrust, albedo, general relativity corrections, and other effects as deemed necessary to achieve the required fidelity of the generated ephemeris. For all but the lowest fidelity ephemeris, gravity contributions to the forces on an Earth-orbiting spacecraft are typically modeled using a gravity field for the Earth and point-mass third-body gravity forces arising from the Moon and Sun.

Let B_i be a celestial body and $\mathbf{R}_{Bi}$ locate its (center-of-mass) position with respect to the solar system barycenter O in inertial axes. Let $\mathbf{R}$ locate the spacecraft S from O in inertial axes. Then the Newtonian equations of motion for S are:

$$m\ddot{\mathbf{R}} = \mathbf{F}_S + \sum_{i=0} GmM_{Bi} \frac{\mathbf{R}_{Bi} - \mathbf{R}}{\left\|\mathbf{R}_{Bi} - \mathbf{R}\right\|^3} \tag{1}$$

where m is the spacecraft mass, G the universal gravitation constant, M_{Bi} is the mass of B_i, and $\mathbf{F}_S$ is the sum of all forces on the spacecraft other than the point-mass gravitational force caused by all the celestial bodies (e.g., additional gravitational forces over the point-mass effect, drag, solar radiation pressure, general relativistic corrections, etc.).

Note that the motions of the celestial bodies are taken to be known as a function of time – the motion of the spacecraft and bodies are not being solved simultaneously. The most precise ephemeredes for planets and moons are available from JPL. JPL's Developmental Ephemeris was derived from a numerical integration of the major bodies of the solar system, using barycenter values for the outer planets. The equations of motion considered general relativistic effects and oblateness of the Earth and Moon (Ref. 1). Thus, the motion of the bodies is not simply driven by point-mass gravity.

Ephemeredes for the moons of other planets are also available from JPL, in SPICE format (Ref. 2), created by investigators using the best knowledge of the force environment at the time. We would expect that oblateness would indeed have a significant effect on the moons of Jupiter and Saturn and be considered in their ephemeredes.

These equations are now transformed to a coordinate system using the same axes whose origin is at the (center-of-mass) position of one of the bodies, say B_0. Let $\mathbf{r}_{Bi}$ and $\mathbf{r}$ be the relative position of B_i and S with respect to B_0, i.e.,

$$\mathbf{r}_{Bi} = \mathbf{R}_{Bi} - \mathbf{R}_0,\ \mathbf{r} = \mathbf{R} - \mathbf{R}_0,\ \text{where } \mathbf{R}_0 \triangleq \mathbf{R}_{B_0} \tag{2}$$

Rewriting (1) using (2) and dividing through by m, we find

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}_S}{m} - \ddot{\mathbf{R}}_0 + \sum_{i=0} \left(GM_{Bi} \frac{\mathbf{r}_{Bi} - \mathbf{r}}{\left\|\mathbf{r}_{Bi} - \mathbf{r}\right\|^3} \right) \tag{3}$$

Rearranging, we find

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{\left\|\mathbf{r}\right\|^3} + \sum_{i>0} \left(\mu_i \frac{\mathbf{r}_{Bi} - \mathbf{r}}{\left\|\mathbf{r}_{Bi} - \mathbf{r}\right\|^3} \right) - \ddot{\mathbf{R}}_0 + \frac{1}{m}\mathbf{F}_S \tag{4}$$

where $\mu = GM_{B0}$ and $\mu_i = GM_{Bi}$. Now the equations of motion for B_0 are

$$M_0\ddot{\mathbf{R}}_0 = \mathbf{F}_0 + \sum_{i>0} G M_0 M_{Bi} \frac{\mathbf{r}_{Bi}}{\left\|\mathbf{r}_{Bi}\right\|^3}, \text{ where } M_0 \triangleq M_{B_0} \tag{5}$$

where $\mathbf{F}_0$ represents all other forces on B_0 other than the point-mass gravity of the other celestial bodies. Combining (4) and (5), we find the equations of motion for the spacecraft, expressed using a frame whose origin is B_0:

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{\left\|\mathbf{r}\right\|^3} + \sum_{i>0} \mu_i \left(\frac{\mathbf{r}_{Bi} - \mathbf{r}}{\left\|\mathbf{r}_{Bi} - \mathbf{r}\right\|^3} - \frac{\mathbf{r}_{Bi}}{\left\|\mathbf{r}_{Bi}\right\|^3} \right) + \frac{1}{m}\mathbf{F}_S - \frac{1}{M_0}\mathbf{F}_0 \tag{6}$$

The first term in (6) is the point-mass effect of gravity of B_0 and the second term is the classical third-body gravity perturbation acceleration. We refer to this as the classical third-body formula because it is widely used and found in astrodynamics texts (e.g. Ref. 3).

The classical third-body gravity formula (i.e., the summation term in (6)) is comprised of two terms: the first term is the direct term that depends on the distance to the third body from the spacecraft; the second term is the indirect term and only depends on the location of the third body relative to B_0. The direct term arises from the forces on the spacecraft while the indirect term arises from the non-inertial nature of the frame being used to express the equations of motion.

In principle, the solutions to (1), (4) and (6) for different reference bodies B_0 will be equivalent (i.e., producing the same trajectory when converted to a common frame). In practice, however, the solutions will be different because of modeling assumptions made in evaluating (6). Three common assumptions are made:

(i) any difference in the μ_i values used for modeling the third-body perturbation in (6) as compared to the actual μ_i values used when computing the ephemeredes for B_0 in (5) is inconsequential;

(ii) only a small set of the celestial bodies need be modeled for the third-body gravitational effect;

(iii) the celestial bodies themselves are dominated by the point-mass gravitational effects so that $\mathbf{F}_0$ can be ignored.

The first assumption can be eliminated if the correct μ_i values are used (i.e., those consistent with the value used for creating the ephemeredes of the celestial bodies). The second assumption follows from an analysis of the third-body gravity terms in (6). It is this third assumption that we explore below.

The magnitude of the disparity of the solutions will be larger in cases where the motion of B_0 differs significantly from point-mass motion. In the solar system, the motion of the major planets derives largely from point-mass motion and so the effect should be small. The motion of moons, however, may deviate from point-mass motion in the presence of planetary oblateness. We would then expect that a larger disparity would occur for spacecraft traveling near a moon.

CHARACTERIZING THIRD-BODY PERTURBATIONS

The effect of truncating the set of celestial bodies is most easily seen in the third-body gravity terms in (6). Whenever $\mathbf{r} << \mathbf{r}_{Bi}$, the direct and indirect terms nearly cancel out no matter their magnitude; thus, those bodies B_i have little effect on the solution to (6). This is the reason that only the Moon and Sun are usually considered for Earth-orbiting spacecraft. In (1), only the direct terms are present; bodies may be ignored only if the direct term itself is sufficiently small. For Earth-orbiting spacecraft, Jupiter (and other bodies at times) must be considered when evaluating (1) to generate accurate ephemeris. (We deem an ephemeris accurate if the inclusion of an additional celestial body does not appreciably change the trajectory over the time span being considered.) In (4), $\ddot{\mathbf{R}}_0$ includes the indirect terms for all the celestial bodies. A direct term must be included in (4) to offset the indirect term in $\ddot{\mathbf{R}}_0$ unless the direct term itself is sufficiently small (in which case that body has little impact on $\ddot{\mathbf{R}}_0$ itself). In making comparisons, we will be mindful of including sufficient celestial bodies to produce accurate ephemeredes.

We will assess the impact of the third assumption on several different test cases by comparing the solutions to (1), (4) and (6) when both $\mathbf{F}_S$ and $\mathbf{F}_0$ are taken as zero. We would expect the solutions to (1) and (4) to always match closely, but not match solutions to (6) when B_0 experienced more than point-mass gravitational effects.

To compute $\ddot{\mathbf{R}}_0$, the acceleration of B_0 in the inertial frame in (4), we rely on numerical central difference formulas since the accelerations of the celestial bodies are not provided directly by JPL. We compare two formulas for obtaining accelerations from velocity data, using a variety of step sizes. (We also tried using position samples but in general this proved less accurate and more sensitive than using velocity formulas.) The central differences formulas are:

$$\begin{aligned}
\ddot{\mathbf{R}}_0(t) &= \frac{\mathbf{V}_0(t+\Delta t)-\mathbf{V}_0(t-\Delta t)}{2\Delta t}+O(\Delta t^2) \\
\ddot{\mathbf{R}}_0(t) &= \frac{-\mathbf{V}_0(t+2\Delta t)+8\mathbf{V}_0(t+\Delta t)-8\mathbf{V}_0(t-\Delta t)+\mathbf{V}_0(t-2\Delta t)}{12\Delta t^2}+O(\Delta t^4)
\end{aligned} \tag{7}$$

where $\mathbf{V}_0$ denotes the velocity of B_0 and Δt denotes the sample step size. Though the formulas are generally more accurate when Δt is small, if Δt is too small they have precision problems in floating point arithmetic due to subtracting numbers that agree to many significant digits.

RESULTS

Earth-Moon System

Six test cases are considered in the Earth-Moon system. Three are orbits about the Earth: a low-earth orbit (LEO), a highly-elliptical orbit (HEO), and a geosynchronous orbit (GEO). Two are orbits about the Moon: a low-lunar orbit (LLO) and an elliptical lunar orbit (ELO). The sixth test case is a five day transfer orbit from the Earth to the Moon (XFER). The transfer starts at low-earth orbit (290 km altitude) just after a trans-lunar injection burn (TLI) and ends at 100 km altitude above the Moon, just before a lunar orbit insertion burn (LOI) would occur. Table 1 gives the orbital elements of the first five test cases and the initial state vector in J2000 of the transfer case. The first five test cases have an epoch of 1 Jul 2007 12:00:00 UTC, and the transfer case has an initial epoch of 1 Jul 2007 05:35:24.178 UTC. The orbits are integrated for five days with a 20 second step size with a Runge-Kutta-Fehlberg 7(8) integrator using STK/Astrogator.

Table 1: Test cases in Earth-Moon system

Test Case	*a* (km)	*e*	*i* (deg)	ω (deg)	Ω (deg)	ν (deg)
LEO	6678.136	0.01	28.5	0	0	0
HEO	26553.4	0.741	63.4	270	0	0
GEO	42164.0	0.0001	1.0	0	0	0
LLO	1837.4	0.01	45	0	0	0
ELO	12000	0.75	45	0	0	0
	X (km)	Y (km)	Z (km)	Vx (km/s)	Vy (km/s)	Vz (km/s)
XFER	-6654.097	437.354	-11.608	-0.642728	-9.537455	5.108033

In addition to the Earth, Moon, and Sun, all planets are included in the integration. Including all planets is necessary when integrating in the barycenter (1) or when using numerical differentiation to compute $\ddot{\mathbf{R}}_0$ in (4), because the direct terms of each planet is significant. When using the classical formula the planets are not significant, because the direct and indirect terms (nearly) cancel out. Table 2 shows the maximum difference in position over five days between including and not including each planet for the LEO test case when integrating in the barycenter and when using the classical formula. The position of the planets is found from JPL DE 405 files, and the gravitational parameters from JPL DE 405 are used in the integration. At this epoch Mercury is closer than Mars to Earth, so it has a larger effect than Mars for the barycenter integration. In the classical integration, only Jupiter and Venus have a significant effect; the other planets only add noise to the results. In the barycenter case the Sun has a larger effect than the Moon, because its gravity is larger, but in the classical case the Moon has a larger effect than the Sun, because the Sun is farther away.

Table 2: Effect of each body on LEO position over 5 days

Planet	Barycenter (m)	Classical (mm)
Sun	6990000	292
Moon	39700	598
Jupiter	427	3.10
Venus	109	1.09
Saturn	34.8	0.0540
Mercury	3.20	0.218
Mars	2.36	0.0181
Uranus	1.75	0.0243
Neptune	0.652	0.0284

To determine the best method for computing $\ddot{\mathbf{R}}_0$, both the second-order and fourth-order central difference formulas in (7) are used with step sizes of 1000, 100, 10, 5, 1, and 0.1 seconds. Using these combinations (4) is integrated in two different reference frames: one with Earth as the origin, and one with the Moon as the origin. Figures 1 through 6 show the maximum difference, in millimeters, between integrating in the different reference frames for both the second-order (O2) and fourth-order (O4) formulas. Because changing the origin should not affect the trajectory if the acceleration of the origin is modeled correctly, the differencing method that yields the least difference between the trajectories should be modeling the acceleration best.

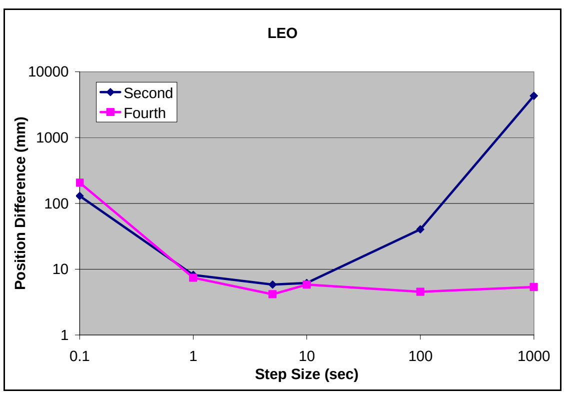

Figure 1: Numerical differentiation position differences in LEO

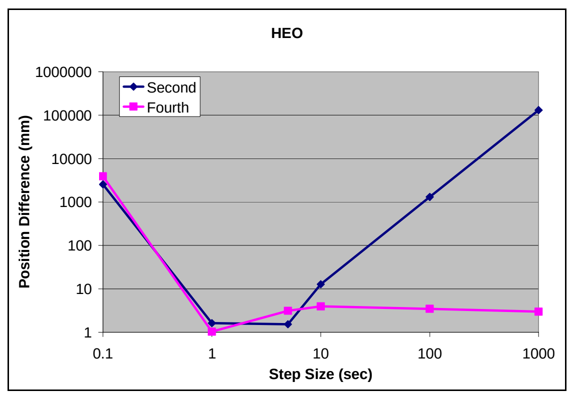

Figure 2: Numerical differentiation position differences in HEO

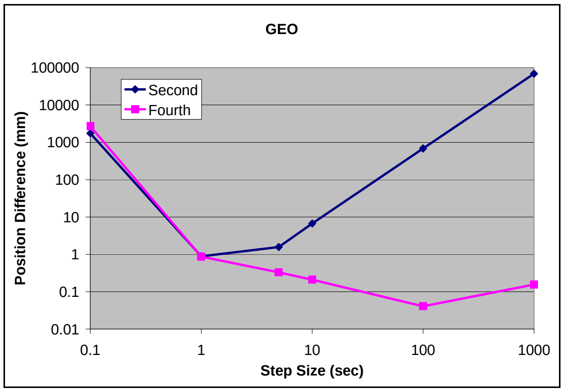

Figure 3: Numerical differentiation position differences in GEO

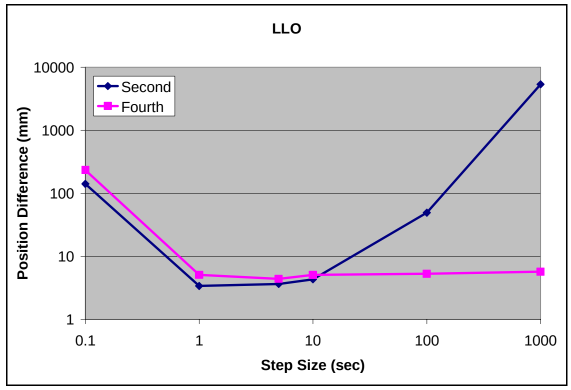

Figure 4: Numerical differentiation position differences in LLO

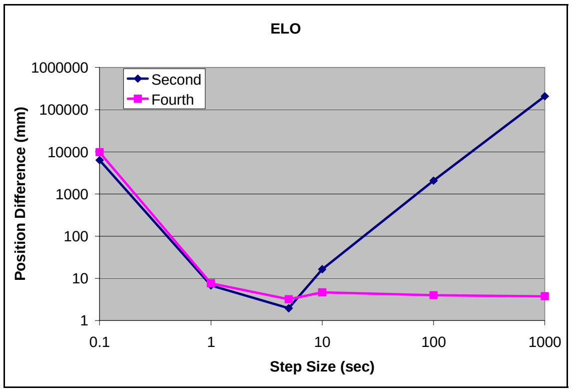

Figure 5: Numerical differentiation position differences in ELO

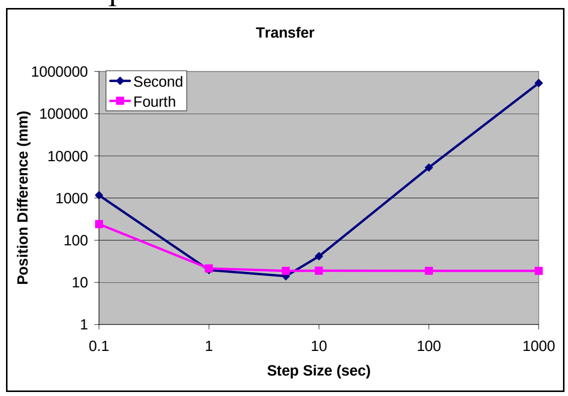

Figure 6: Numerical differentiation position differences for Transfer

The figures show that the fourth-order formula is less sensitive to step size than the second order formula, which is expected because it uses more samples. The differences decrease as the step size decreases in the second-order case down to a step size of 5 sec, while in the fourth-order case the differences are relatively constant for step sizes between 1000 sec and 1 sec. For both formulas the differences increase at a step size of 0.1 seconds, where precision problems occur from subtracting close numbers. Some test cases show an increase in the difference at 1 sec compared to 5 sec while the others show similar results at 5 sec and 1 sec. Because we wish to have one method that works well for all test cases, we use a five second step size with the fourth-order formula in the comparisons against the solar system barycenter and classical approaches. Though not shown in the figures, using the Earth-Moon barycenter as the origin gives similar differences to those shown in Figures 1-6 for the combinations of step size and central differencing formula, when compared to using the Earth or the Moon as the origin. When the Earth-Moon barycenter is used in the numerical approach, its acceleration with respect to the solar system barycenter is used as $\ddot{\mathbf{R}}_0$ in (4).

To measure the effect of neglecting $\mathbf{F}_0$ in (6), the test cases are integrated using the barycenter (1), using the numerical approach (4) with the fourth-order formula and a 5 sec step size, and using the classical approach (6). For the numerical and classical approaches, the integrations are performed using three origins: the Earth, the Moon, and the Earth-Moon barycenter. In the Earth-Moon barycenter case, the classical formula has no indirect term for the Earth and the Moon, only for the Sun and other planets. Table 3 shows the maximum differences over 5 days for each test case between the using either the numerical or classical approaches for each origin and the using the solar system barycenter.

Table 3: Maximum differences against integrating in the solar system barycenter in Earth-Moon system (meters)

	Classical			Numerical		
Test case	Earth	Moon	Barycenter	Earth	Moon	Barycenter
LEO	0.288	0.833	0.668	0.124	0.123	0.118
HEO	7.07	39.0	30.5	0.408	0.406	0.419
GEO	3.310	9.53	9.29	0.0492	0.0489	0.0486
LLO	0.328	1.36	0.875	0.195	0.190	0.189
ELO	6.97	28.8	21.1	0.0174	0.0147	0.0124
XFER	33.9	61.3	150	0.699	0.718	0.711

The table shows meter-level differences in the classical approach compared to using the solar system barycenter. Using the Moon as the origin with the classical approach gives more difference than using the Earth, which is more likely due to the Earth's oblateness effects on the Moon's orbit than the Moon's oblateness effects on the Earth's orbit, so the classical approach models the Earth better. Using the Earth-Moon barycenter as the origin in the classical approach gives less difference in most cases than using the Earth, but more than the Earth.

Using the numerical approach matches the barycenter better than the classical approach, in some cases by an order of magnitude or more. This is expected because the equations of motion used in the numerical approach (4) come directly from the equations for the barycenter approach (1) with no assumptions. In fact, the differences shown in Table 3 for the numerical approach are most likely errors in the barycenter integration. The position of the spacecraft with respect to the barycenter is very large, on the order of 1.5×10^{8} km. Using numerical integration with numbers this large gives a significant amount of round-off error, which will build up over the integration. Because the numerical approach provides consistent results regardless of origin to 10 mm or better (Figures 1-6), it can be considered to be more accurate than using the solar system barycenter as the origin.

Jovian system

A similar study is performed in the Jovian system with a test case orbit about Io. Table 5 gives the orbital elements for the orbit. For this test the Sun, Jupiter, Io, Europa, Ganymede, Europa, Callisto, and all other planets are included in the force model, as well as Earth's moon. The position of the Sun, planets and moons is found from JPL Spice files (Ref. 2), and the gravitational parameters used in creating the Spice files are used for the bodies. For Saturn, Uranus and Neptune, the barycenter positions from the Spice files are used, as are the system gravitational parameters.

Table 5: Test case in Jovian system (orbit about Io)

r_p (km)	e	i (deg)	ω (deg)	Ω (deg)	ν (deg)
2500	0.05	45	0	0	0

As in the Earth-Moon tests, both central difference formulas (7) are used with step size of 1000, 100, 10, 5, 1 and 0.1 seconds. These formulas are used with three origins: Jupiter, Io, and the barycenter of the Jovian system. Using the Jovian barycenter as the origin gives the same results as using Jupiter for all step sizes for both formulas. Figure 7 shows the maximum differences over five days between using Jupiter and Io as the origin for each formula and step size. The figure shows that the fourth-order case gives relatively constant results for step sizes between 100 sec and 1 sec, while the second-order case shows the difference decreasing between 1000 sec and 1 sec. For both formulas the cases match best at 1 sec. However, for consistency with the Earth-Moon test, a step size of 5 seconds is used in the comparisons against the other approaches.

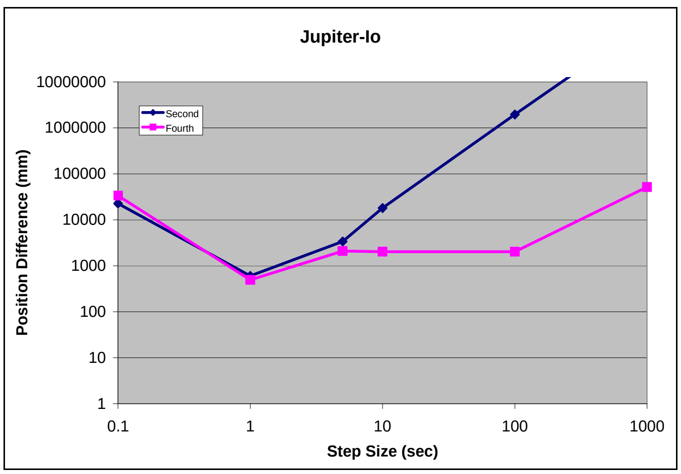

Figure 7: Numerical differentiation position differences in Jupiter-Io case

Table 6 shows the differences between the classical, numerical, and barycenter approaches. The table shows that using Io with the classical approach gives hundreds of kilometers of difference. When the numerical approach is used, the difference between using Io as the origin and using the solar system barycenter is reduced to meters. The numerical approach also reduces the difference in using Jupiter as the origin versus using the solar system barycenter, by an order of magnitude. Using the Jupiter barycenter in the classical approach gives the same result as using the Jupiter barycenter, or Jupiter itself, in the numerical approach.

Table 6: Maximum differences against integrating in the solar system barycenter for Jupiter-Io case (meters)

Classical			Numerical		
Jupiter	Io	Barycenter	Jupiter	Io	Barycenter
45.7	447000	1.08	1.08	3.05	1.08

Saturn system

A study is also performed in the Saturn system with an orbit about Enceladus. Table 7 gives the elements of the orbit. In this test the Sun, Saturn, Enceladus, all planets and Earth's moon are included in the force model, as well as Saturn's other moons: Mimas, Tethys, Dione, Rhea, Titan, Hyperion, Iapetus, and Phoebe. JPL Spice files are used for the ephemeris of the Sun, planets and moons, and the gravitational parameters used in the Spice files are also used (Ref. 2). The barycenter position and system gravitational parameters are used for Jupiter, Uranus and Neptune. Because the orbit around Enceladus cannot be maintained due to Saturn's gravity, the test case is integrated for only 12 hours.

Table 7: Test case in Saturn system (orbit about Enceladus)

a (km)	e	i (deg)	ω (deg)	Ω (deg)	ν (deg)
363.35	0.067	86	274	66	220

Figure 8 shows the maximum differences over 12 hours between trajectories computed using (4) with Saturn and Enceladus as origins with both central difference formulas (7) for various step sizes. The figure shows similar results as Figures 1-7, and shows the best agreement using a 5 second step size with the fourth-order formula.

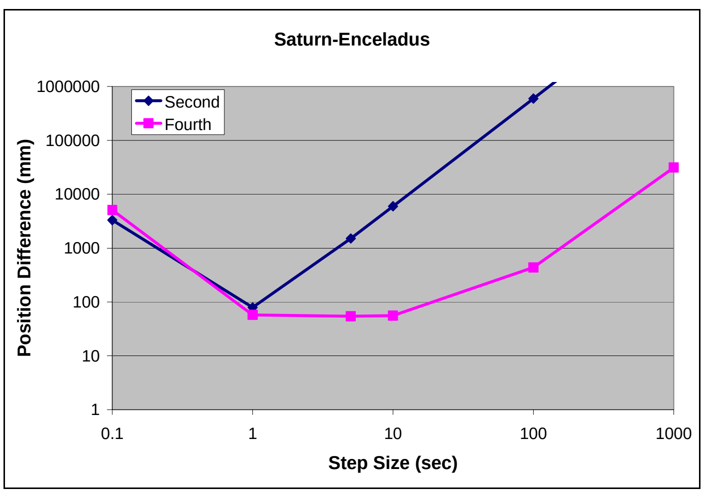

Figure 8: Numerical differentiation position differences in Saturn-Enceladus case

Table 8 shows the differences between the classical, numerical, and barycenter approaches. As in the case about Io, using Enceladus as the origin with the classical approach gives hundreds of kilometers of difference, in this case after only 12 hours.

With the numerical approach the difference is improved by six orders of magnitude. Using Saturn as the origin in the classical approach also shows meter-level differences against the solar system barycenter. The smallest differences arise from using the Saturn barycenter, which matches the solar system barycenter to 0.186 m using both the classical and numerical approaches.

Table 8: Maximum differences against integrating in the solar system barycenter for Saturn-Enceladus case (meters)

Classical			Numerical		
Saturn	Enceladus	Barycenter	Saturn	Enceladus	Barycenter
7.66	221000	0.186	0.189	0.234	0.186

DISCUSSION

The test cases demonstrate that the solutions using different reference bodies are essentially the same when using (1) and (4), but may differ when using (6). The difference is most pronounced when the reference body's ephemeris was generated using more than point-mass gravitational effects. At first, one may believe that this makes the use of (6) suspect; however, the reverse is true: the actual trajectory produced by (6) is a better model of the ephemeris of the spacecraft than all the others.

Consider again the use of Io as the reference body. The ephemeris for Io includes the J2 gravitational effect of Jupiter. However, by setting $\mathbf{F}_S$ to zero, Jupiter's J2 effect is not considered when modeling the forces on the spacecraft. This is a modeling error: the J2 effect may be modeled as being on or off, but needs to be consistently applied to all bodies, the spacecraft included. Since the ephemeris for Io has already made the modeling decision to include Jupiter's J2 effect, the effect must be included on the spacecraft as well. Thus, testing using $\mathbf{F}_S$ as zero doesn't model the spacecraft motion properly.

Even so, the solution already created using (6) with Io as the reference body better reflects the actual spacecraft motion because it can be viewed as incorporating Jupiter's J2 effect (at least approximately). We computed (6) viewing both $\mathbf{F}_S$ and $\mathbf{F}_0$ as zero. However, they need not be viewed as being individually zero: it is their difference (divided by the appropriate masses) that was actually treated as zero. When modeling Jupiter's J2 effect, the difference expression in (6) becomes

$$\mathbf{F}_T \triangleq \frac{1}{m}\mathbf{F}'_S - \frac{1}{M_o}\mathbf{F}_0 = \mu_J \left(\Phi(\mathbf{r} - \mathbf{r}_{Bi}) - \Phi(-\mathbf{r}_{Bi}) \right) \tag{8}$$

where $\mathbf{F}'_S$ represents the component of $\mathbf{F}_S$ caused by the non-point mass gravitation of B_0 and $\Phi(\cdot)$ represents the expression of Jupiter's J2 effect normalized by its gravitational coefficient. When the spacecraft is near Io, $\mathbf{r} << \mathbf{r}_{Bi}$, (8) is approximately zero. Thus, the solutions we created about Io to (6) are consistent with a model incorporating Jupiter's J2 effect on both the spacecraft and Io. As this represents a consistent modeling of the gravitational effect of Jupiter on all bodies, this trajectory betters models the actual spacecraft motion. Of course, an even better model would actually evaluate (8) and not treat it as zero---this would require the ability to evaluate the gravitational field of Jupiter even when treating it as a third body.

CONCLUSIONS

While in principle the formulation of the equations of motion has little bearing on the obtained trajectories, in practice this is not the case. Different formulations may lead to different trajectories because of assumptions being made when implementing the formulations. Above all, one must be careful to be consistent when including gravitational effects in each formulation to achieve accurate trajectories.

The formulation with the origin at the solar system barycenter, equation (1), suffers from numerical truncation problems for spacecraft that orbit near a celestial body. Moreover, the gravitational effects from all celestial bodies must be included on the spacecraft to achieve an accurate trajectory (including non-point mass effects). The formulation that uses a reference celestial body as the origin but computes the reference body acceleration numerically, equation (4), eliminates the truncation issue but still must model all gravitational effects from all celestial bodies on the spacecraft.

There are several advantages to the last formulation, equation (6), which separates the reference body acceleration into two parts. First, the dominant point-mass gravitational effect of third bodies only appears as a tidal term (i.e., in the form of a difference of a direct and indirect term). When the spacecraft remains close to the reference body, the direct and indirect terms nearly cancel. This provides a mechanism for ignoring certain celestial body gravitational effects entirely while still maintaining an accurate trajectory. A consequence of this, however, is that if the tidal force $\mathbf{F}_T$ is ignored (i.e., treated as zero) but is not nearly zero, solutions to (6) using different reference bodies will be different. Only by computing $\mathbf{F}_T$, by including the full gravity fields of all the bodies, will solutions to (6) be equivalent using different reference bodies.

REFERENCES

1. Newhall, X.X., Standish, Jr., E.M., and Williams, J.G. “DE 102: A numerically integrated ephemeris of the Moon and planets spanning forty-four centuries.” *Astronomy and Astrophysics*. Vol. 125. pp. 150-167. 1983.
2. JPL Spice files available at: ftp://naif.jpl.nasa.gov/pub/naif/generic_kernels/spk/.
3. Vallado. *Fundamentals of Astrodynamics and Applications.* Second Edition. p. 542. Microcosm Press, El Segundo, CA. 2004.